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On the Strong Solvability of a Nonlocal Boundary Value Problem for the Laplace Equation in an Unbounded Domain

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Abstract. In this work a nonlocal problem for the Laplace equation in an unbounded domain is considered. The notion of a strong solution of this problem is defined. Using the Fourier method, we prove the correct solvability of the considered problem in Sobolev spaces generated by a weighted mixed-norm. This problem in the classical formulation was previously considered by E. I. Moiseev [1]. The same type of problem was considered in the work of M. E. Lerner and O. A. Repin [2].

Key Words and Phrases: Laplace equation, nonlocal problem, weighted Sobolev space, strong solution

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1. Introduction

Consider the following nonlocal boundary value problem (so far formal) for a degenerate elliptic equation:

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$$y^m u_{xx} + u_{yy} = 0\,, \quad m > -2\,, \quad 0 < x < 2\pi\,, \quad y > 0\,, \tag{1}$$

$$u\,(x, 0) = f\,(x)\ ,\ \ u\,(0, y) = u\,(1, y)\ \ , \tag{2}$$

$$u_x\,(0, y) = 0\ ,\ \ 0 < x < 2\pi\ ,\ \ y > 0\ . \tag{3}$$

This problem is nonlocal, and the support boundary condition are semi-infinite lines, and one of them has a normal derivative. Such problems exhibit specific

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features compared to those with local boundary conditions. Earlier, F.I. Frankl [3]; [4, p. 453-456] considered a problem with a nonlocal boundary condition for a mixed type equation.

The Bitsadze-Samarskii problem [5] for elliptic equations is also nonlocal with supports on a part of the boundary of the domain and, moreover, the supports are free from other boundary conditions. In the work of N.I. Ionkin and E.I. Moiseev [6], for multidimensional parabolic equations, a boundary value problem was solved with a nonlocal condition of the form (2), was supported by the characteristic and improper parts of the domain boundary. Problem (1)–(3) in the classical formulation was considered in [1] and [2].

In this article, we consider problem (1)-(3) in the case $m = 0$, i.e. for the Laplace equation in a weighted Sobolev space with a weight from the Mackenhoupt class. The notion of a strong solution of this problem is defined. The correct solvability of this problem is proved by the Fourier method. It should be noted that the study of the solvability of elliptic equations in weighted Sobolev spaces encounters certain difficulties in comparison with the non-weighted case. Therefore, very few works are devoted to this direction (see, e.g. [12, 13, 14, 15]) and the development of the corresponding theory is still far from being completed.

We define the notion of a strong solution for this problem. The correct solvability of the problem is proved using the Fourier method. It should be noted that the study of the solvability of elliptic equations in weighted Sobolev spaces encounters certain difficulties compared to the non-weighted case. As a result, very few works have been devoted to this direction (see, e.g., [12, 13, 14, 15]), and the development of the corresponding theory is still far from complete.

2. Auxiliary facts and notation

We will use the following standard notation.$N-$ will be the set of natural numbers, $\alpha = (\alpha_1; \alpha_2) \in Z_+ \times Z_+ -$ will be the multiindex, where $Z_+ = \{0; 1; ...\}$. Accept $\partial^\alpha u = \frac{\partial^{|\alpha|} u}{\partial x^{\alpha_1} \partial y^{\alpha_2}}$, where $|\alpha| = \alpha_1 + \alpha_2$. $|M| -$ will stand for the Lebesgue measure of the set M; $\bar{M}-$is a closure of the set M; $C^\infty\left(\bar{M}\right) -$are infinitely differentiable functions on $\bar{M}$; $C_0^\infty(M) -$ are infinitely differentiable and compactly supported functions in M. Throughout this paper we will assume that p' is the number conjugate to p, $1 < p < +\infty$: $\frac{1}{p'} + \frac{1}{p} = 1$.

Let us define the weighted Sobolev space we need. Let $\nu : [0, 2\pi] \to (0, +\infty) -$ be some weight function, i.e. $\left|\nu^{-1}\{0; +\infty\}\right| = 0$. In the sequel, we will assume that the weight $\nu(\cdot)$ satisfies $\nu(2\pi - x) = \nu(x)$, a.e. for $x \in (0, 2\pi)$ and and that it is periodically continued along the real axis R with a period of 2π.

Denote by $L_{p;\nu}(\Pi)$ the Banach function space on $\Pi = (0,\ 2\pi) \times (0, +\infty)$ with mixed norm

$$\|f\|_{L_{p;\nu}(\Pi)} = \int_0^{+\infty} \left(\int_0^{2\pi} |f(x;y)|^p \nu(x)\, dx \right)^{\frac{1}{p}} dy \ , \quad 1 \le p < +\infty.$$

Sobolev space generated by the norm

$$\|u\|_{W^2_{p;\nu}} = \sum_{|\alpha| \le 2} \|\partial^\alpha u\|_{L_{p;\nu}(\Pi)} ,$$

denote by $W^2_{p;\nu}(\Pi)$. We also denote by $L_{p;\nu}(I)$, where $I = (0,\, 2\pi)$, the weighted Lebesgue space with the norm

$$\|f\|_{L_{p;\nu}(I)} = \left(\int_I |f(x)|^p \nu(x)\, dx \right)^{\frac{1}{p}} .$$

We also consider a weighted Sobolev space $W^2_{p;\nu}(I)$ on I, generated by the norm

$$\|f\|_{W^2_{p;\nu}(I)} = \|f\|_{L_{p;\nu}(I)} + \|f'\|_{L_{p;\nu}(I)} + \|f''\|_{L_{p;\nu}(I)} .$$

We need the class of Muckenhoupt weights $A_p(I)$. This is the class of weights that satisfy the condition

$$\sup_{J \subset I} \left(\frac{1}{|J|} \int_J \nu(t)\, dt \right) \left(\frac{1}{|J|} \int_I |\nu(t)|^{-\frac{1}{p-1}} \right)^{p-1} < +\infty,$$

where sup is taken over all intervals $J \subset I$ and $|J| -$ is a length of the interval J.

In obtaining the main results, we will use the fact that the classical trigonometric system is a basis in weighted Lebesgue spaces. From the results of the work of R.A. Hunt, W.S. Young [16], it follows that the following proposition is true.

Proposition 1. *The trigonometric system* $\{1; \cos nx; \sin nx\}_{n \in N}$ *forms a basis for the weighted Lebesgue space* $L_{p;\nu}(I)$, $1 < p < +\infty \ \Leftrightarrow \ \nu \in A_p(I)$.

It should be noted that similar questions regarding perturbed trigonometric systems were studied in [17, 18, 19].

From $\nu \in A_p(I)$, $1 < p < +\infty$, it follows that a continuous embedding $L_{p,\nu}(I) \subset L_1(I)$ holds. As a result, it is clear that the continuous embedding $W^2_{p;\nu}(\Pi) \subset W^2_1(\Pi)$ is also true. Therefore, every function $u \in W^2_{p;\nu}(\Pi)$ has a trace $u/_{\partial\Pi}$ on the boundary $\partial\Pi$ (well-defined with respect to the Lebesgue measure on $\partial\Pi$).

So let's accept the following

Definition 1. *We will say that a function* $u \in W^2_{p;\nu}(\Pi)$ *is a strong solution of problem* (1)–(3) *if equality* (1) *is satisfied a.e.* $(x;y) \in \Pi$, *and its trace* $u/_{\partial\Pi}$ *satisfies relations* (2), (3).

Let us introduce into consideration the following systems of functions

$$\{1;\ \cos nx;\ x\sin nx\}_{n\in N}, \tag{4}$$

$$\{u_0(x)\ ;\ u_n(x)\ ;\ v_n(x)\}_{n\in N}, \tag{5}$$

where

$$u_0(x) = \frac{1}{2\pi^2}(2\pi - x)\,; u_n(x) = \frac{1}{\pi^2}(2\pi - x)\cos nx; v_n(x) = \frac{1}{\pi}\sin nx\,,\ n \in N.$$

In obtaining the main result, we will essentially use the following

Theorem 1. *Let* $\nu \in A_p(I)$, $1 < p < +\infty$. *Then the system* (4) *forms a basis for* $L_{p;\nu}(I)$.

Proof. The dual space of $L_{p;\nu}(I)$ is $L_{p';\nu}(I)$. It is quite obvious that the system (5) belongs to $L_{p';\nu}(I)$ and, moreover, it is biorthonormalized with the system (4) (see [1]). This implies that system (4) is minimal in $L_{p;\nu}(I)$. Let us prove that it is also complete in $L_{p;\nu}(I)$. It is obvious that if $\nu \in A_p(I)$, $1 < p < +\infty$, then $C_0^\infty(I)$ is dense in $L_{p;\nu}(I)$. Indeed

$$\nu \in A_p(I)\ \Rightarrow\ \exists\ p_0 \in (1, +\infty)\ \Rightarrow\ \nu \in L_{p_0}(I)\,.$$

Regarding this fact, one can see e.g. [7, 9, 10]. We have

$$\int_I |f|^p\,\nu\,dx \le \left(\int_I |f|^{pp_0'}\,dx\right)^{\frac{1}{p_0'}} \left(\int_I \nu^{p_o}dx\right)^{\frac{1}{p_0}}.$$

Hence it follows that $L_{p_1}(I) \subset L_{p;\nu}(I)$, where $p_1 = pp_0'$, i.e. it is valid

$$\|f\|_{L_{p;\nu}(I)} \le C\,\|f\|_{L_{p_1}(I)}\,,$$

where $C > 0$ is a constant, independent of f. Since $C_0^\infty(I)$ is dense in $L_{p_1}(I)$, then it follows from the previous inequality that it is also dense in $L_{p;\nu}(I)$.

Let $f \in C_0^\infty(I)$ be an arbitrary function. Put $g = \frac{1}{\pi^2}(2\pi - x)\,f(x)$. It is clear that $g \in C_0^\infty(I)$. We have

$$f_n^+ = \frac{1}{\pi^2}\int_0^{2\pi} f(x)\,(2\pi - x)\,\cos nx\,dx = \int_0^{2\pi} g(x)\,\cos nx\,dx =$$

$$= \frac{1}{n} \int_0^{2\pi} g'(x) \sin nx \, dx = -\frac{1}{n^2} \int_0^{2\pi} g''(x) \cos nx \, dx \ \Rightarrow$$

$$\Rightarrow \ \left| f_n^+ \right| \leq \frac{c}{n^2} \ , \ \ \forall n \in N.$$

Similarly, it is established that

$$\left| f_n^- \right| \leq \frac{c}{n^2} \ , \ \ \forall n \in N,$$

where

$$f_n^- = \frac{1}{\pi} \int_0^{2\pi} f(x) \sin nx \, dx.$$

Consequently, the series

$$F(x) = \frac{1}{2\pi} \int_0^{2\pi} (2\pi - x) \ f(x) \ dx + \sum_{n=1}^{\infty} \left(f_n^+ \ \cos nx + f_n^- x \sin nx \right), \qquad (6)$$

converges uniformly on I. Since, according to the results of [1], system (4) forms a basis for $L_2(I)$, it is clear that $F = f$. It follows from the uniform convergence that the series (6) converges to f and in $L_{p;\nu}(I)$. This implies the completeness of system (4) in $L_{p;\nu}(I)$.

Let us prove the basicity of system (4) in $L_{p;\nu}(I)$. Consider projectors

$$S_{n;m}(f) = \sum_{k=0}^{n} (u_k; \ f) \cos kx + \sum_{k=1}^{m} (v_k; \ f) \, x \sin kx, \forall \ n \geq 0 \ ; \ \forall \ m \geq 1,$$

where the denotation

$$(g \, ; \ f) = \int_0^{2\pi} f(x) \, g(x) \, dx,$$

is accepted. We have

$$(u_0; \ f) = \left(\frac{1}{2\pi}; F \right), (u_k; \ f) = \left(\frac{1}{\pi^2} \cos kx; \ F \right), \forall \ k \in N;$$

where $F(x) = (2\pi - x) f(x)$. Taking into account these relations, we obtain

$$\|S_{n;m}(f)\|_{L_{p;\nu}(I)} \leq \left\| \left(\frac{2\pi - x}{2\pi}; F \right) + \sum_{k=1}^{n} \left(\frac{2\pi - x}{\pi^2} \cos kx, \ f \right) \cos kx \right\|_{L_{p;\nu}(I)} +$$

$$+ \left\| x \sum_{k=1}^{m} \left(\frac{1}{\pi} \sin kx; \ 4\pi f \right) \sin kx \right\|_{L_{p;\nu}(I)} \leq$$

$$\leq /\, from\ the\ basicity\ \{1;\ \cos nx;\ \sin nx\}\ in\ L_{p;\nu}(I)\ /\leq$$

$$\leq c\left(\|F\|_{L_{p;\nu}(I)} + 2\pi\,\|4\pi F\|_{L_{p;\nu}(I)}\right) \leq c\,\|f\|_{L_{p;\nu}(I)}\ ,\ \forall\, n \geq 0\,;\ \forall\, m \geq 1,$$

where $c > 0$ is an absolute constant (may be different in different places). Consequently, the projectors $\{S_{n;m}\}$ are uniformly bounded, and as a result, it follows from the basicity criterion that the system (4) forms a basis for $L_{p;\nu}(I)$. Theorem is proved.

In obtaining the main results, we will essentially use the following Minkowski inequality (see, e.g., [20, p. 24]), for the integrals.

Proposition 2. *[20]. Let $(M_k;\, \sigma_{M_k};\, \mu_k)$, $k = \overline{1,2}$- be measurable space with σ-finite measure μ_k and $F(x;y)$ $\mu_1 \times \mu_2$ be a measurable function. Then*

$$\left\|\int_{M_1} F(x;y)\, d\mu_1(x)\right\|_{L_p(\mu_2)} \leq \int_{M_1} \|F(x;\cdot)\|_{L_p(\mu_2)}\, d\mu_1(x)\,,$$

where

$$\|f\|_{L_p(\mu)} = \left(\int |f|^p\, d\mu\right)^{\frac{1}{p}}.$$

3. Main results

Let us first take the notation $\Gamma_0 = \{(0;\, y) : \forall\, y > 0\}$, and $\Gamma_{2\pi} = \{(2\pi;\, y)\ ,\ \forall\, y > 0\}$. Consider the following nonlocal problem

$$\Delta u = 0\,,\ \ (x;y) \in \Pi, \tag{7}$$

$$u/_I = f\ ,\ \ u/_{\Gamma_0} = u/_{\Gamma_{2\pi}}\ ,\ \ u_x/_{\Gamma_0} = 0. \tag{8}$$

Under the solution of this problem we mean a function $u \in W^2_{p;\nu}(\Pi)$, satisfying a.e. in Π the equality (7) and with respect to the traces on the boundary $\partial\Pi = I \bigcup \Gamma_0 \bigcup \Gamma_{2\pi}$ of which relations (8) hold. First we prove the uniqueness of the solution. It is true the following

Theorem 2. *Let $\nu \in A_p(I)$, $1 < p < +\infty$, and $f \in W^2_{p;\nu}(I)$ & $f(2\pi) = f(0) = f'(0) = 0$. Then if the problem* (7), (8) *has a solution in $W^2_{p;\nu}(\Pi)$, then it is unique.*

Proof. Let $u \in W^2_{p;\nu}(\Pi)$ be some solution of the problem (7), (8). Let $y > 0$ be an arbitrary number and assume $\Gamma_y = \{(x;y)\ :\ x \in I\}$. It is clear that $u \in W^2_1(\Pi)$. The trace of a function on Γ_y is denoted by $u_y\ :\ u_y = u/_{\Gamma_y}$. Let us show that $u_y \in L_{p,\nu}(I)$. Put $\Pi_y = \{(x;\xi)\ :\ x \in I\ \&\ \xi \in (0,y)\}$. It is obvious that $u \in W^2_{p;\nu}(\Pi_y)$. It follows from the condition $w \in A_p(I)$ that $w \in A_p(\Pi_y)$ (it is verified directly) and as a result $C^\infty(\bar{\Pi}_y)$ is dense in $W^2_{p;\nu}(\Pi_y)$. Let's first assume that $u \in C^\infty(\bar{\Pi}_y)$. Without loss of generality, we assume that $u(x;0) = 0,\ \forall\, x \in I$. Consequently

$$u_y(x) = u(x;y) = \int_0^y \frac{\partial u(x;\xi)}{\partial \xi} d\xi, \quad \text{a.e. } x \in I.$$

Applying here the Minkowski inequality (Proposition 2), we obtain

$$\|u_y\|_{L_{p;\nu}(I)} \le \left\| \frac{\partial u}{\partial y} \right\|_{L_{p;\nu}(\Pi_y)} \le \|u\|_{W^2_{p;\nu}(\Pi_y)} .$$

Using this estimate and the fact that $C^\infty(\bar{\Pi}_y)$ is dense in $W^2_{p;\nu}(\Pi_y)$, it is proved in exactly the same way as in the non-weighted case that the trace $\forall\, u \in W^2_{p;\nu}(\Pi)$ on Γ_y satisfies the estimate

$$\|u_y\|_{L_{p;\nu}(I)} \le \|u\|_{W^2_{p;\nu}(\Pi)} .$$

If u satisfies (7), then it is known that $u \in C^\infty(\Pi) \Rightarrow u_y(x) = u(x;y),\ \forall x \in I$. Suppose $u(\cdot;\cdot)$ is a solution to problem (7), (8). Consider

$$\left.\begin{aligned} u_0(y) &= \tfrac{1}{2\pi}\textstyle\int_0^{2\pi} u(x;y)\,(2\pi - x)\,dx\ , \\ u_n(y) &= \tfrac{1}{\pi^2}\textstyle\int_0^{2\pi} u(x;y)\,(2\pi - x)\,\cos nx\,dx\ , \\ v_n(y) &= \tfrac{1}{\pi}\textstyle\int_0^{2\pi} u(x;y)\,\sin nx\,dx\ ,\ \ n \in N\ . \end{aligned}\right\} \tag{9}$$

It is quite obvious that for a.e. $x \in I$ it holds

$$u(x;\,y+h) - u(x;y) = \int_y^{y+h} \frac{\partial u(x;t)}{\partial t} dt\ ,\ \forall\, y > 0\,.$$

Since $\frac{\partial u}{\partial y} \in L_1(\Pi)$, it follows from Theorem 1.1.1 of the monograph [8, p.13] that the functions $\{u_n;\, v_n\}$ are twice differentiable and can be differentiated under the integral sign. Considering that u satisfies equation (7), multiplying it by $\sin nx$ and integrating over I, for $v_n(\cdot)$ we obtain the following relation

$$v_n''(y) - n^2 v_n(y) = 0\ ,\ y > 0\,. \tag{10}$$

Let $\alpha \in C^\infty(R)$ be such that $\alpha(y) \equiv 1$ in a sufficiently small neighborhood of the point $y = 0$ and $\alpha(y) = 0,\ \forall y : |y| \ge 1$. Considering the function

$F(x;y) = \alpha(y)\ u(x;y)$, we get $F(x;y) = 0$, $\forall\ y \geq 1$. Therefore, in the following calculations, we will assume that $u(x;y) = 0,\ \forall\ y \geq 1$. So, we have

$$u(x;y) = -\int_y^1 \frac{\partial u(x;\xi)}{\partial \xi}\, d\xi, \quad \text{a.e.} \quad x \in I \Rightarrow$$

$$\Rightarrow \quad f(x) = u(x;0) = -\int_0^1 \frac{\partial u(x;\xi)}{\partial \xi}\, d\xi, \quad \text{a.e. } x \in I.$$

Consequently

$$|u(x;y) - f(x)| \leq \int_0^y \left| \frac{\partial u(x;\xi)}{\partial \xi} \right| d\xi, \ \text{ a.e. } x \in I.$$

From this it immediately follows

$$\int_I |u(x;y) - f(x)|\ dx \leq \int_I \int_0^y \left| \frac{\partial u(x;\xi)}{\partial \xi} \right| d\xi\, dx.$$

Since $|\{(x;\xi) : (x;\xi) \in I \times (0,y)\}| \to 0\,,\ y \to +0$, then it is clear that $u_y(\cdot) \to f(\cdot)\ ,\ y \to +0$, in $L_1(I)$. It is easy to see that $\nu_n(\cdot) \in W_1^2(0,+\infty)$. Hence it follows that $\exists \lim\limits_{y \to +0} \nu_n(y) = \nu_n(0)\ ,\ \forall n \in N$. It follows directly from these two relations that

$$\nu_n(0) = \frac{1}{\pi} \int_0^{2\pi} f(x) \sin nx\, dx\ ,\ \ \forall\ n \in N, \tag{11}$$

is true. On the other hand

$$v_n(y) - v_n(0) = \frac{1}{\pi} \int_0^{2\pi} (u(x;y) - u(x;0))\ \sin nx\, dx =$$

$$= \frac{1}{\pi} \int_0^{2\pi} \int_0^y \frac{\partial u(x;t)}{\partial t} \sin nx\, dt\, dx \quad \Rightarrow$$

$$\Rightarrow \quad |v_n(y) - v_n(0)| \leq \frac{1}{\pi} \iint_\Pi \left| \frac{\partial u}{\partial y} \right| dx\, dy < +\infty.$$

From this it immediately follows

$$\sup_{y>0} |v_n(y)| < +\infty. \tag{12}$$

The solution to problem (10)-(12) is

$$v_n(y) = \frac{1}{\pi} \int_0^{2\pi} f(x) \sin nx\, dx\, e^{-ny}, \forall\ n \in N. \tag{13}$$

Similarly for u_n we obtain

$$u_0(y) = \frac{1}{2\pi}\int_0^{2\pi} f(x)\,(2\pi - x)\,dx,$$

$$u_n(y) = \frac{1}{\pi^2}\int_0^{2\pi} f(x)\,(2\pi - x)\,\cos nx\,dx\,e^{-ny} + \frac{1}{\pi}\int_0^{2\pi} f(x)\,\sin nx\,dx\,ye^{-ny}, \forall\, n \in N. \tag{14}$$

These relations directly imply the uniqueness of the problem. In fact, if $f = 0$, then it is clear that $u_0(y) = u_n(y) = \nu_n(y) = 0$, $\forall\, y > 0$, $\forall\, n \in N$. As $u_y \in L_{p,\nu}(I)$, $\forall\, y > 0$, then from the basicity of the system (4) in $L_{p,\nu}(I)$ follows that $u_y(x) = 0$, a.e. $x \in I$ and $\forall\, y > 0$. Hence it follows that $u(x; y) = 0$, a.e. $(x; y) \in \Pi$. Consequently, the homogeneous problem has only a trivial solution, and this proves its uniqueness.

Theorem is proved.

Let us prove the existence of a solution. The following theorem is true.

Theorem 3. *Let $\nu \in A_p(I)$, $1 < p < +\infty$, the boundary function f satisfies the conditions*

$$f \in W^2_{p;\nu}(I)\ \&\ f(0) = f(2\pi) = f'(0) = 0.$$

Then problem (7), (8) has a (unique) solution in $W^2_{p;\nu}(\Pi)$ and moreover it is valid the following estimate

$$\|u\|_{W^2_{p;\nu}(\Pi)} \le c\,\|f\|_{W^2_{p;\nu}(I)},$$

where $c > 0$ is a constant independent of f.

Proof. Consider the function

$$u(x; y) = u_0(y) + \sum_{n=1}^{\infty} \left(u_n(y)\,\cos nx + v_n(y)\,x\sin nx\right),\ (x; y) \in \Pi,$$

where the coefficients $u_0(\cdot)$, $u_n(\cdot)$, $\nu_n(\cdot)$, $n \in N$, are defined by expressions (13), (14). Let us show that the function u belongs to $W^2_{p;\nu}(\Pi)$. First, consider the series

$$u_1(x; y) = \sum_{n=1}^{\infty} v_n(y)\,x\,\sin nx.$$

We have (formally differentiating term by term)

$$\frac{\partial^2 u_1}{\partial y^2} = \sum_{n=1}^{\infty} v_n''(y)\,x\,\sin nx,$$

$$\frac{\partial u_1}{\partial x} = \sum_{n=1}^{\infty} v_n(y) \sin nx + \sum_{n=1}^{\infty} n v_n(y)\ x \cos nx,$$

$$\frac{\partial^2 u_1}{\partial x^2} = 2\sum_{n=1}^{\infty} n v_n(y) \cos nx - \sum_{n=1}^{\infty} n^2 v_n(y)\ x \sin nx.$$

Denote

$$w(x;y) = \sum_{n=1}^{\infty} n^2 v_n(y)\ x \sin nx.$$

Let us show that the function $w(\cdot;\cdot)$ belongs to $L_{p;\nu}(\Pi)$. Set

$$f_n = \frac{1}{\pi}\int_0^{2\pi} f(x) \sin nx\, dx\,,$$

consequently

$$v_n(y) = f_n e^{-ny}\ ,\ \ n \in N.$$

We have

$$f_n = -\frac{1}{\pi n}\int_0^{2\pi} f(x)\ d\cos nx = -\frac{1}{\pi n}\left(f(2\pi) - f(0) - \int_0^{2\pi} f'(x) \cos nx\, dx \right) =$$

$$= \frac{1}{\pi n}\int_0^{2\pi} f'(x) \cos nx\, dx = \frac{1}{\pi n^2}\int_0^{2\pi} f''(x) \sin nx\, dx = \frac{1}{n^2} f''_n.$$

Thus

$$w(x;y) = \sum_{n=1}^{\infty} f''_n x \sin nx\, e^{-ny}.$$

It is known that if $\nu \in A_p(I)$, then $\exists\ \delta > 0\ :\ \nu \in L_{1+\delta}(I)$ (see e.g. [10, p. 395]). Let $\alpha = 1+\delta\,,\ \frac{1}{\alpha} + \frac{1}{\alpha'} = 1$. Applying the Hölder inequality, we obtain

$$\int_0^{2\pi} |w\ (x;y)|^p\, \nu(x)\, dx \le c \left(\int_0^{2\pi} |w\,(x;y)|^{p\alpha'}\, dx \right)^{\frac{1}{\alpha'}},$$

where c is a constant independent of $w\ (\cdot;\cdot)$. Consider the following cases.

I. $p > 2$. Consequently $p_1 = p\alpha' > 2$. Applying the classical Hausdorff-Young theorem (see e.g. [21, p. 154]) from the previous inequality, we have

$$\left(\int_0^{2\pi} |w\ (x;y)|^p\, \nu(x)\, dx \right)^{\frac{1}{p}} \le c \left(\int_0^{2\pi} |w\ (x;y)|^{p_1}\, dx \right)^{\frac{1}{p_1}} \le$$

$$\leq c\left(\sum_{n=1}^{\infty}\left|f_n''e^{-ny}\right|^{p_1'}\right)^{\frac{1}{p_1'}}\leq c\sum_{n=1}^{\infty}\left|f_n''e^{-ny}\right|.$$

From this it immediately follows

$$\|w\|_{L_{p;\nu}(\Pi)}\leq c\sum_{n=1}^{\infty}\left|f_n''\right|\int_0^{+\infty}e^{-ny}dy=c\sum_{n=1}^{\infty}\frac{|f_n''|}{n}.$$

Consequently

$\|w\|_{L_{p;\nu}(\Pi)}\leq c\left(\sum_{n=1}^{\infty}|f_n''|^{\beta}\right)^{\frac{1}{\beta}}\leq$/ the Hausdorff-Young inequality /

$$\leq c\left\|f''\right\|_{L_{\beta'}(I)}, \tag{15}$$

where $\beta\geq 2$ is some number, $\frac{1}{\beta}+\frac{1}{\beta'}=1$. It is known that if $\nu\in A_p(I)$, $1<p<+\infty$, then $\exists\, q:\ 1<q<p\ \Rightarrow \nu\in A_q(I)$. Let $r=\frac{p}{q}\ \Rightarrow\ 1<r<p$. We have

$$\int_I|g|^r\,dx=\int_I|g|^{\frac{p}{q}}\,\nu^{\frac{1}{q}}\,\nu^{-\frac{1}{q}}dx\leq\left(\int_I|g|^p\,\nu\,dx\right)^{\frac{1}{q}}\left(\int_I\nu^{-\frac{q'}{q}}\,dx\right)^{\frac{1}{q'}}.$$

Taking into account $-\frac{q'}{q}=\frac{1}{1-q}$, from $\nu\in A_q(I)$ follows that $\nu^{-\frac{1}{q-1}}\in L_1(I)$. Then it follows from the previous inequality that $f\in L_r(I)$ and

$$\|g\|_{L_r(I)}\leq c\,\|g\|_{L_{p;\nu}(I)},$$

holds, where $c>0-$is a constant independent of g. It is quite obvious that continuous embedding $L_{p,\nu}(I)\subset L_{\alpha}(I)$ is true for $\forall\,\alpha\in(1,r)$. Let's take β so large that $1<\beta'<r$ is fulfilled. Then from (15) we get

$$\|w\|_{L_{p;\nu}(\Pi)}\leq c\left\|f''\right\|_{L_{p;\nu}(I)}.$$

II. $p\in(1,2)$ Following the expression $\alpha=1+\delta$, we choose $\delta>0$ so small that $p_1=p\,\alpha'>2$ holds (this is possible because of $\alpha\to 1+0\Rightarrow\alpha'\to+\infty$). Based on this inequality, the further reasoning is carried out in a completely analogous way to case I.

Other series in the expression $u(\cdot;\cdot)$ are evaluated in a similar way and as a result we get

$$\|u\|_{W_{p;\nu}^2(\Pi)}\leq c\left\|f''\right\|_{L_{p;\nu}(I)},$$

where $c>0$ is a constant independent of f. That $u(\cdot;\cdot)$ satisfies equation (7) is verified directly.

Let's check the fulfillment of the boundary conditions. The trace operators on Γ_0 ; $\Gamma_{2\pi}$ and I, will be denoted by θ_0 ; $\theta_{2\pi}$ and θ_I, respectively. Let us show that $\theta_I u = f$. From the boundedness of the operator $\theta_I \in \left[W^2_{p;\nu}(\Pi) \, ; \, L_{p;\nu}(I)\right]$ it follows that if $u_m \to u$ in $W^2_{p;\nu}(\Pi)$, then $u_m/_I \to u/_I$ in $L_{p;\nu}(I)$.

So, consider the following functions

$$u_m(x;y) = u_0(y) + \sum_{n=1}^{m} \left(u_n(y) \cos nx + v_n(y)\, x \sin nx\right) \,, \quad (x;y) \in \Pi, m \in N.$$

We have

$$\theta_I u_m = u_m(x;0) = u_0(0) + \sum_{n=1}^{m} \left(u_n(0) \cos nx + v_n(0)\, x \sin nx\right) =$$

$$= \frac{1}{2\pi} \int_0^{2\pi} f(x)(2\pi - x)\, dx +$$

$$+ \sum_{n=1}^{m} \left(\frac{1}{\pi^2} \int_0^{2\pi} f(x)(2\pi - x) \cos nx dx \cos nx + \frac{1}{\pi} \int_0^{2\pi} f(x) \sin nx dx \, x \sin nx \right).$$

From the basicity of the system (4) in $L_{p;\nu}(I)$ it follows that $\theta_I u_m \to f$, $m \to \infty$, in $L_{p;\nu}(I)$ and as a result $\theta_I u = f$.

Consider the operators θ_0 and $\theta_{2\pi}$. It is absolutely clear that $\theta_0 u_m = \theta_{2\pi} u_m$, $\forall\, m \in N$. It is obvious that $\theta_0 u_m \to \theta_0 u$ and $\theta_{2\pi} u_m = \theta_{2\pi} u \;\Rightarrow\; \theta_0 u = \theta_{2\pi} u$. Thus, the boundary conditions (8) are satisfied.

Theorem is proved.

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