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On Solvability Of One Boundary Value Problem For Laplace Equation in Banach-Hardy Classes

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Abstract. In this paper we consider the Dirichlet problem for the Laplace equation in Hardy classes generated by an additive-invariant Banach function space on the unit circle. It is shown that the classical Dirichlet problem for the Laplace equation has a unique solution for every boundary function from the considered space. It is considered a boundary problem for the Laplace equation with oblique derivatives in the Hardy classes generated by separable subspaces of rearrangement-invariant spaces in which the infinitely differentiable functions are dense. Noetherness of this problem is established and the index of this problem is calculated.

Key Words and Phrases: Banach function space, additive-invariant, Laplace equation, oblique derivatives, Noetherness, Hardy class

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1. Introduction

The solvability theory of elliptic equations in the classical setting (for Hölder classes $H^{\alpha}(\Omega)$ and Lebesgue spaces L_p) has been extensively investigated and fundamental monographs by many mathematicians have been devoted to its study (see e.g. [1], [3], [14], [15], [21], [22], [23]). Recently, various problems for elliptic equations have been considered for different Banach function spaces (e.g. $L_{p,w}$-weighted Lebesgue spaces, L_{Φ}-Orlicz spaces, $L_{p(.)}$-variable Lebesgue spaces, $L_{p)}$-grand Lebesgue space and rearrangement-invariant Banach function spaces on bounded domains) and this tradition continues today (see, for example, [5], [6], [7], [8], [9],[10], [11], [12], [13], [16], [24], [25], [26], [27], [28], [29], [30], [31], [32], [33]).

In general, to use classical methods as grand Lebesgue spaces, it is necessary to choose a separable subspace in which infinitely differentiable functions are dense. In [6], [7], [8], [9], [10], [11], [12], [13], [17], [18] such separable subspaces of some Banach function spaces are defined.

In [5] the Dirichlet problem for the classical Laplace equation with oblique derivatives in the Hardy classes formed by the weighted Lebesgue space on a unit disk is considered. Noetherness of the considered problem is established and its index is calculated.

This paper is a continuation and generalization of the work [5]. We consider the problem in the case of additive-invariant (in particular, in rearrangement-invariant) spaces defined on the unit circle and Hardy spaces on the unit disk generated by these spaces. In general the Banach function spaces under consideration are non-separable, some adjustments need to be made to apply the classical method. For this purpose, we introduce a special separable subspace by using the additive shift operator $(T_\delta f)(x) = f(x+\delta)$, and consider the corresponding problem with respect to this subspace where the set of compactly supported infinitely differentiable functions is dense. We show that the classical Dirichlet problem for the Laplace equation has a unique solution for every boundary function from the considered space in the additive-invariant case. We also consider a special boundary problem for the Laplace equation with oblique derivatives in the Hardy classes generated by a separable subspace of rearrangement-invariant spaces. We establish the Noetherness of the problem and calculate the index of this problem.

Throughout this paper we will use the following standard notations: N will denote the set of all positive integers, $Z_+ = \{0\} \cup N$, Z set of all integer numbers, C the set of complex numbers, R the set of all real numbers, $D = \{z \in C : |z| < 1\}$ the unit disk, and $T = \{z : |z| = 1\}$ the unit circle. $|E|$ will denote linear Lebesgue measure of a set $E \subset T$, and $C_0^\infty[-\pi, +\pi]$ the set of compactly supported infinitely differentiable functions defined on $(-\pi, +\pi)$.

We identify the arbitrary function $f : T \to C$ with the function defined on $[-\pi; \pi)$ as follows

$$f : [-\pi; \pi) \to C(orR) \Leftrightarrow f(t) := f\left(e^{it}\right),$$

and we assume that it is extended periodically to all of R: $f(t+2\pi) = f(t)$. For $f : D \to R$, we consider the following family of functions

$$f_r(t) = f\left(re^{it}\right),\ 0 \le r < 1,\ t \in [-\pi; \pi).$$

$L_p(T) \equiv L_p(-\pi, \pi),\ 1 \le p < +\infty$, is a classical Lebesgue space of measurable functions with the norm

$$\|f\|_p = \left(\int_{-\pi}^{\pi} |f|^p dt\right)^{\frac{1}{p}}.$$

Let $A(D)$ denote the class of all analytic functions on the unit disk D. $H(D)$ denote the class of all harmonic functions on the unit disk, i.e.

$$H(D) = \{u : D \to R : \Delta u = 0 \ in \ D\}.$$

Let $X(T)$ be a Banach function space on T with a Lebesgue measure and a norm $\|.\|_X$. By $B_X(a, r), a \in X, r > 0$, denote the open ball centered at a, with radius r. X^* will denote the dual space, $X' \subset X^*$ the associated space with norm $\|.\|_{X'}$, $X_a(T)$ will be the subspace of absolutely continuous functions, and $X_b(T)$ the closure of all bounded functions. Let $A \subset X(T)$ be some subset. Then $L[A]$ will denote the linear span of A.

Let X be a metric space, and let $A \subset X$ and $B \subset X$ be subsets, then $dist_X(A, B)$ denotes the distance between these subsets.

2. Hardy Spaces Generated by Additive-Invariant Banach Function Spaces

2.1. Needful Information

In this subsection we define the spaces and classes of measurable functions to be used throughout this paper. We also establish some of their properties, which imply that the classical Dirichlet problem has a unique solution. For further information about Lebesgue spaces, Banach function spaces, Hardy classes the reader can refer to [2], [4], [19], [20].

Banach Function Space. Let $(M, \mathcal{M}, \mu)$ be a measure space, F denotes the set of all measurable functions whose values belong to $[-\infty; +\infty]$, F^+ is the cone of μ -measurable functions whose values lie in $[0; +\infty]$, χ_E denotes the characteristic function of a μ -measurable subset $E \in \mathcal{M}$. F_0 denotes the class of functions which are finite μ-a.e. F_s denotes the collection of all simple functions.

Definition 1. *A mapping $\rho : M \to [0; +\infty]$, is called a Banach function norm if, for $\forall f, g, f_n \in F^+, \forall a \geq 0, \forall E \in \mathcal{M}$, the following properties hold:*

(P1) $\rho(f) = 0 \Leftrightarrow f = 0 \ \mu - a.e. \ \rho(af) = a\rho(f), \ \rho(f + g) \leq \rho(f) + \rho(g)$;

(P2) $g \leq f \ \mu - a.e. \ \Rightarrow \rho(g) \leq \rho(f)$;

(P3) $0 \leq f_n \uparrow f \Rightarrow \rho(f_n) \uparrow \rho(f)$;

(P4) $\mu(E) < +\infty \Rightarrow \rho(\chi_E) < +\infty$;

(P5) $\mu(E) < +\infty \Rightarrow \int_E f d\mu \leq C_E \rho(f)$, *where C_E is some positive number may be depend on E and ρ, but doesn't depend of f.*

Let ρ be a function norm on F^+. The collection $X = X(\rho)$ of all functions in F for which $\rho(|f|) < +\infty$, is called a Banach function space. For each $f \in X$ the norm is defined as $\|f\|_X = \rho(|f|)$.

Let ρ be a function norm, its associate norm $\rho^{'}$ is defined on F^{+} by

$$\rho'(g) = \sup\left\{\int_M fgd\mu : \rho(f) \le 1\right\}.$$

Definition 2. *Let ρ be a function norm. The space $X = X(\rho)$ is a corresponding Banach function space, $\rho^{'}$ is the corresponding associate norm. Then the Banach function space determined by the associate norm $\rho^{'}$ is called the associate of X, denoted as $X^{'} = X^{'}(\rho')$.*

We will also use Fatou's lemma and the Minkowski-type inequality to obtain many results.

Lemma 1. (Fatou's Lemma ([2], [4])) *Let $X = X(\rho)$ be a Banach function space and $f_n \in X$. $(n = 1, 2, \ldots)$ If $f_n \to f$ $\mu-$a.e. and $\lim_{n\to\infty} \inf \|f_n\|_X < \infty$, then $f \in X$ and*

$$\|f\|_X \le \|f_n\|_X .$$

Minkowski-type inequality (see, [17]). Let $\Omega_1 \subset R^n, \Omega_2 \subset R^k$ *are some domains, $X(\Omega_1)$-Banach function space, $f : \Omega_1 \times \Omega_2 \to R$is a measurable function. If $f(.,y) \in X(\Omega_1)$ for $m-$a.e. $y \in \Omega_2$ and $\|f(.,y)\|_X \in L_1(\Omega_2)$, then the following inequality holds.*

$$\left\|\int_{\Omega_2} f(x,y)dy\right\|_X \le \int_{\Omega_2} \|f(.,y)\|_X \, dy.$$

In Banach function spaces, the following analogous of Hölder's inequality holds.

Hölder's inequality. *Let X be a Banach functional space and $X^{'}$ is a corresponding associate space. Then for $\forall f \in X$, $\forall g \in X^{'}$, it follows that fg is integrable and*

$$\int |fg|d\mu \le \|f\|_X \|g\|_{X'}.$$

Let $f \in X(\Omega)$ be some function. Then

$$\|f\|_{X(\Omega)} = \sup\left\{\left|\int_\Omega fgd\mu\right| : g \in X'(\Omega), \|g\|_{X'(\Omega)} \le 1\right\}.$$

Rearrangement-invariant Banach function space. *Let $f \in F_0$. The distribution function of f is the following function defined on $[0;\infty)$:*

$$\mu_f(\lambda) = \mu\{x \in M : |f(x)| > \lambda\}.$$

Two functions $f, g \in F_0$, whose distribution functions are the same, are called equimeasurable, i.e. $\mu_f(\lambda) = \mu_g(\lambda)$ for all $\lambda \ge 0$.

Definition 3. *Let $(M, \mathcal{M}, \mu)$ be a totally σ - finite measure space. If for every pair of equimeasurable functions, $g \in F_0^+$, the equality $\rho(f) = \rho(g)$ holds, then the norm ρ is called a rearrangement-invariant norm. The Banach function space generated by a rearrangement-invariant norm is called a rearrangement-invariant Banach function space.*

Boyd's indices. *Let $X = X(\rho)$ be a rearrangement-invariant Banach function space over a measure space $(R^n,\ \mu)$, where μ is a Lebesgue measure in R^n. For each $t > 0$, let E_t denote the dilation operator defined on $F_0\,(R^+, m)$ (m is linear Lebesgue measure in R^+) by*

$$(E_t f)\,(s) = f(ts),\ (0 < t < \infty).$$

Let

$$h_X(t) = \left\| E_{1/t} \right\|_{[\widetilde{X}]},\ (0 < t < \infty).$$

Definition 4. *Let $X = X(\rho)$ be a rearrangement-invariant Banach function space over a measure space $(R^n,\ \mu)$. The Boyd indices of X are the numbers α_X and β_X defined by*

$$\alpha_X = \sup_{0<t<1} \frac{\log h_X(t)}{\log t},\quad \beta_X = \sup_{1<t<\infty} \frac{\log h_X(t)}{\log t}.$$

Additive-Invariant Banach Function Space ([14]). *Let $X\,(R^n)$ be some Banach function space. We say that it is an additive-invariant Banach function space if for $\forall \delta \in R,\ \forall f \in X,$ the following holds $\|f(.+\delta)\|_X = \|f(.)\|_X$.*

In the case $\Omega \subset R^n$, we assume that every function is extended by zero on R^n. Thus, in this case, the shift operator is defined as follows.

$$(T_\delta f)\,(x) = \begin{cases} f(x+\delta) & ,\, x+\delta \in \Omega, \\ 0 & ,\ x+\delta \notin \Omega. \end{cases}$$

The subspace $X_s(\Omega)$ is defined as the set of all functions $f \in X(\Omega)$ for which the shift operator is continuous, i.e.

$$X_s(\Omega) = \left\{ f \in X(\Omega) : \|f(.+\delta) - f(.)\|_X \quad \delta \to 0 \quad \to 0 \right\}.$$

Accept the following property.

Property (β)**:** *For $\forall E_n \to \varnothing$, with $E_n \subset \Omega$ it follows that $\|\chi_{E_n}\|_{X(\Omega)} \to 0$.*

Remark 1. i) *In the series of works* [9], [12], [13], [17], *it is proven that if the additive-invariant space (particularly rearrangement-invariant space) has Property* (β)*, then the relation*

$$X_s(\Omega) = X_a(\Omega) = X_b(\Omega) = \overline{C_0^\infty(\Omega)},$$

holds true, where $\Omega \subset R^n$ *is a bounded domain.*

ii) *It is clear that if* $\Omega \subset R^n$ *is a bounded domain, then the following inclusions are true*

$$L_\infty(\Omega) \subset X(\Omega) \subset L_1(\Omega).$$

In the sequel we will only consider the case $X(T) = X(-\pi;\pi)$ with the Lebesgue measure. We will deal with the following spaces and classes of measurable (complex or real-valued) functions defined on the unit circle and disk.

The space $h_X(D)$ is defined by the relation

$$h_X(D) = \left\{ u \in H(D) : \sup_{0 \le r < 1} \|u_r(.)\|_X < \infty \right\},$$

and with the norm $\|u\|_{h_X(D)} = \sup_{0 \le r < 1} \|u_r(.)\|_{X(D)}$.

In the case $X = L_p$, we will use the notation $h_p(D)$. The corresponding **Hardy-Sobolev spaces** $h_X^1(D)$ *and* $h_{X_s}^1(D)$ are defined by the relations

$$h_X^{(1)}(D) = \left\{ u \in h_X : \frac{\partial u}{\partial r}, \frac{\partial u}{\partial \phi} \in h_X \right\}; \quad h_{X_s}^{(1)}(D) = \left\{ u \in h_{X_s} : \frac{\partial u}{\partial r}, \frac{\partial u}{\partial \phi} \in h_{X_s} \right\},$$

and with the norm

$$\|u\|_{h_X^{(1)}(D)} = \|u\|_{h_X(D)} + \left\| \frac{\partial u}{\partial r} \right\|_{h_X(D)} + \left\| \frac{\partial u}{\partial \phi} \right\|_{h_X(D)},$$

where (r, ϕ) denotes the polar coordinates. These spaces are Banach spaces.

Hardy class $H_X^+(D)$ is the corresponding class of analytic functions, i.e.

$$H_X^+(D) = \left\{ f \in A(D) : \|f\|_{H_X^+(D)} = \sup_{0 < r < 1} \|f_r(.)\|_X < \infty \right\}.$$

It is clear that $H_X^+(D) = \{ f \in A(D) : f \ \in h_{X(D)} \}$.

The following statement is valid.

Statement 1. *Let* $X(T)$ *be a Banach function space. If there exist* $p \ge 1$ *such that* $X(T) \subset L_p(T)$, *then the embeddings*

$$h_X(D) \subset h_p(D) \qquad \text{and} \qquad H_X^+ \subset H_p^+(D),$$

are valid.

2.2. Some properties

In this subsection we will establish some properties of above introduced spaces and classes.

Theorem 1. *Let $X(T)$ be a Banach function space. Then:*

a) *For $\forall f \in h_X(D)$, there is a non-tangential boundary value function $f^+ \in X(T)$, and the relation*

$$f_r(\phi) = f(r,\phi) = \frac{1}{2\pi}\int_{-\pi}^{\pi} P_r(\phi-\theta) f^+(\theta) d\theta, \tag{1}$$

is valid, where $P_r(\phi) = \frac{1-r^2}{1-2r\cos\ \phi\ +r^2}$ is the Poisson kernel with respect to the unit disk. Moreover, the inequality

$$\left\| f^+ \right\|_X \leq \|f\|_{h_X}, \tag{2}$$

holds.

b) *$\forall u \in H_X(D)$ has a non-tangential boundary value function $u^+ \in X(T)$ and the Cauchy formula*

$$u(z) = \frac{1}{2\pi i}\int_T \frac{u^+(\xi)}{\xi - z} d\xi, \tag{3}$$

is valid.

Proof. Statement 1 implies that for $\forall f \in h_X$, we have $f \in h_1$. Therefore, by classical theorems, it follows that $\lim_{r\to 1} f_r = f^+$ *in* L_1. Consequently, for $\{r_n\} \subset (0;1)$: $r_n \to 1-0$, there exist $f_{r_n} \to f^+ \in L_1$ *a.e. as* $r_n \to 1^-$. By definition of $h_X(D)$, the sequence $\{f_{r_n}\}_n$ is bounded in $X(T)$. Then by Fatuo's lemma we obtain $f^+ \in X(T)$, and

$$\left\| f^+ \right\|_X \leq \sup_n \|f_{r_n}\|_X \leq \sup_r \|f_r\|_X = \|f\|_{h_X}.$$

Taking into account that every function from $X(T)$ belongs to $L_1(T)$ and that the representations (1) and (3) are valid for functions from $L_1(T)$, we complete the proof of the theorem. The theorem is proved. ■

Let us define the trace operator in Hardy classes: $\gamma : h_X \to X(T)$, such that $\gamma(u) = u^+ \in X(T)$, $\gamma : h_X \to X(T)$, where u^+ is non-tangential boundary function for $u \in h_X$.

From Theorem 1 it follows that *Range* $\gamma \subset X(T)$. The following theorem shows that in the case of an additive-invariant space, *Range* $\gamma = X(T)$ is valid.

Theorem 2. *Let $X(T)$ be an additive-invariant Banach function space. Then $\gamma\left(h_X(D)\right) = X(T)$ and for $\forall u \in h_X(D)$ the relation*

$$\|u\|_{X(D)} \leq \|\gamma u\|_{X(T)}\,, \tag{4}$$

is valid.

Proof. Let $f \in X(T)$ and consider the Poisson-Lebesgue integral

$$u_r(\phi) = u(r,\phi) = \frac{1}{2\pi}\int_{-\pi}^{\pi} P_r(\phi-\theta) f(\theta) d\theta, \qquad \forall re^{i\phi} \in D.$$

We have

$$\|u_r\|_{X(T)} = \frac{1}{2\pi} \sup_{\|g\|_{X'(T)}\leq 1} \left| \int_{-\pi}^{\pi} u_r(\phi) g(\phi) d\phi \right|.$$

Let us estimate this integral. Using Fubini's theorem and Hölder's inequality we obtain

$$\begin{aligned}\frac{1}{2\pi}\left|\int_{-\pi}^{\pi} u_r(\phi)g(\phi)d\phi\right| =& \frac{1}{2\pi}\left|\int_{-\pi}^{\pi}\left(\int_{-\pi}^{\pi} P_r(\phi-t)f(t)dt\right)d\phi)\right| = \\ =& \frac{1}{2\pi}\left|\int_{-\pi}^{\pi} P_r(t)\left(\int_{-\pi}^{\pi} f(\phi-t)g(\phi)d\phi\right)dt\right| \leq \\ \leq& \frac{1}{2\pi}\int_{-\pi}^{\pi} P_r(t)\,\|f\|_{X(T)}\,\|g\|_{X'(T)}\,dt \leq \|f\|_{X(T)}\,.\end{aligned}$$

Therefore

$$\|u_r\|_{X(T)} \leq \|f\|_{X(T)}\,, \forall r \in [0,1),$$

and

$$\|u\|_{h_X(D)} \leq \|f\|_{X(T)}\,.$$

Consequently $\gamma^{-1} f \in h_X(D)$. The theorem is proved. ■

Corollary 1. *Let $X(T)$ be an additive-invariant Banach function space. Then:*
i) $\gamma \in [h_X(D); X(T)]$ *is an isometric isomorphism;*
ii) $h_X(D)$ *is a Banach space;*
iii) *If* $f \in X(T)$ *and* $0 \leq r_1 < r_2 < 1$, *then* $\|f_{r_1}\|_X \leq \|f_{r_2}\|_X$;
iv) *If* $f \in X(T)$, *then* $\|f_r\|_{X(T)} \uparrow \|f^+\|_{X(T)}$, *as* $r \to 1$.

Proof. i) From estimates (2) and (4), it follows that for $\forall f \in X(T)$, if $u\left(re^{it}\right) = P_r * f$, then $\gamma u = f$ *and* $\|u\|_{h_X(D)} = \|f\|_{X(T)} = \|\gamma u\|_{X(T)}$, i.e., γ is an isometric operator.

ii) The statement ii) follows directly from i).

iii) Let $0 \le r_1 < r_2 < 1$. Consider the new function F on T defined by $F(t) = f(r_2, t)$. It is clear that $f(r_1, t) = P_{\frac{r_1}{r_2}} * F$. Consequently, by (4) we have $\|f(r_1, .)\|_{X(T)} \le \|F(.)\|_{X(T)} = \|f(r_2, .)\|_{X(T)}$. iv) The statement iv) follows from iii).

The corollary is proved. ■

Theorem 3. *Let $X(T)$ be an additive-invariant Banach function space and $f^+ \in X_s(T)$. Then* $\lim_{r\to 1^-} \|f_r(t) - f^+(t)\|_{X(T)} = 0$.

Proof. We have

$$f\left(re^{i\phi}\right) = \frac{1}{2\pi}\int_{-\pi}^{\pi} P_r(\phi - t) f^+(t)dt = \frac{1}{2\pi}\int_{-\pi}^{\pi} f^+(\phi - t) P_r(t)dt,$$

and

$$f\left(re^{i\theta}\right) - f^+(\theta) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \left(f^+(\theta - t) - f^+(\theta)\right) P_r(t)dt.$$

Applying the Minkowski inequality we have

$$\left\|f\left(\tau e^{i\theta}\right) - f^+(\theta)\right\|_X \le \frac{1}{2\pi}\int_{-\pi}^{\pi} P_\tau(t) \left\|f^+(\theta - t) - f^+(\theta)\right\|_X dt \le$$
$$\le \left\|f^+(\theta - t) - f^+(\theta)\right\|_X \to 0 \ , \ as \ t \to 0 \ .$$

The theorem is proved.■

From this theorem we obtain the following corollary.

Corollary 2. *Let X be an additive-invariant Banach function space and $f^+ \in X(T)$, $f \in h_X(D)$ and $\gamma f = f^+$. Then the following statements are equivalent:*

i) $f^+ \in X_s(T)$;

ii) $\|f_r(.) - f^+(.)\|_{X(T)} \to 0$, *as* $r \to 1$;

iii) $\|f_r(. + \delta) - f_r(.)\|_{X(T)} \to 0$, *uniformly for* $0 \le r < 1$, *as* $\delta \to 0$.

Proof. i) ⇒ ii): This implication follows directly from Theorem 3.

iii)⇔ i): This equivalence follows from the following relation

$$\left\|f^+(. + \delta) - f^+(.)\right\|_{X(T)} = \lim_{r\to 1^-} \|f_r(. + \delta) - f_r(.)\|_{X(T)}$$

ii)⇒i):It is clear that $f_r(t) \in C(T) \subset X_s(T)$ and $\lim_{r\to 1} f_r(.) = f^+(.)$ in $X(T)$. By the closeness of the subspace $X_s(T)$ in $X(T)$, we have $f^+ \in X_s(T)$.

The corollary is proved. ■

In the sequel we will consider the following spaces

$$h_{X_s}(D) = \left\{ f \in h_X(D) : \gamma f = f^+ \in X_s(T) \right\},$$

and

$$H_{X_s}(D) = \left\{ f \in H_X(D) : \gamma f = f^+ \in X_s(T) \right\}.$$

In the case $X(T) = L_p(T)$, it is clear that $h_{X_s}(D) = h_p(D)$. Consider the following Dirichlet problem

$$\left.\begin{array}{r} \Delta u = 0, \;\; in\ D, \\ \gamma u = f, \;\; on\ T. \end{array}\right\} \tag{5}$$

By solution of this problem we mean the function $u \in h_X(D)$, for which $(\gamma u)(\xi) = f(\xi)$, *a.e.* $\xi \in T$.

From Theorem 2 we have the following corollary

Corollary 3. *Let $X(T)$ be an additive-invariant Banach function space. Then for $\forall f \in X(T)$ the Dirichlet problem (5) has a unique solution in $h_X(D)$, and moreover, $\|u\|_{h_X} = \|f\|_X$.*

Now, we study some properties of the Hardy-Sobolev spaces.

Corollary 4. *Let $X(T)$ be a Banach function space.*

a) If $\frac{\partial u}{\partial r} \in h_X(D)$, then $u \in h_X(D)$;

b) Let $X(T)$ be an additive-invariant Banach function space. If $\frac{\partial u}{\partial r} \in h_{X_s}(D)$, then $u \in h_{X_s}(D)$.

Proof. a) Suppose $\frac{\partial u}{\partial r} \in h_X(D)$. Let us show that $u \in h_X(D)$. We have

$$u(r,\phi) - u(0,0) = \int_0^r \frac{\partial u(\rho,\phi)}{\partial \rho} d\rho.$$

Then by the Minkowski inequality we obtain

$$\|u(r,.) - u(0,0)\|_{X(T)} \le \int_0^1 \left\| \frac{\partial u(\rho,\phi)}{\partial \rho} \right\|_{X(T)} d\rho \le \quad \left\| \frac{\partial u}{\partial \rho} \right\|_{h_X(D)} .$$

Therefore

$$\|u(r,.)\|_{X(T)} \le \left\| \frac{\partial u}{\partial r} \right\|_{h_X(D)} + |u(0,0)|.$$

Since $\frac{\partial u}{\partial r} \in h_X(D)$, it follows that $u \in h_X(D)$.

b) Let $\frac{\partial u}{\partial r} \in h_{X_s}(D)$, Let us show that $u \in h_{X_s}(D)$.

By the definition of $h_{X_s}(D)$, we have

$$\sup_{0\le\rho<1}\left\|\frac{\partial\left(u(\rho, . +\delta)-u(\rho,\ .)\right)}{\partial\rho}\right\|_{X(T)} \to 0\ ,\quad as\quad \delta\to 0.$$

By similar arguments

$$\|u(r, . +\delta)-u(r, .)\|_{X(T)} \le \left\|\frac{\partial\left(u(\rho,\ . +\delta)-u(\rho,\ .)\right)}{\partial\rho}\right\|_{h_X(D)} \to 0,\quad as\quad \delta\to 0.$$

From Corollary 2-iii), it follows that $u \in h_{X_s}(D)$. The corollary is proved. ■

Corollary 5. *Let $X(T)$ be an additive-invariant-Banach function space and let the system $\{u_n(t)\}_{n\in Z_+} \subset X_s(t)$ forms a basis in $X_s(T)$. Then the system $\{u_n(r,t)\}_{n\in Z_+}$ forms a basis in $h_{X_s}(D)$, where*

$$u_n(r,t)=\frac{1}{2\pi}\int_{-\pi}^{\pi} P_r(t-\theta)\ u_n(\theta)d\theta.$$

Proof. Taking into account Corollary 1, Theorem 3 and Corollary 2 we obtain that the relation

$$dist_{h_{X_s}(D)}\left(u_k(r,t), L\left[\{u_n(r,t)\}_{n\ne k}\right]\right) = dist_{X_s(T)}\left(u_k(t), L\left[\{u_n\}_{n\ne k}\right]\right) > 0,$$

is valid, which implies that the system $\{u_n(r,t)\}_{n\in Z_+}$ is minimal in $h_{X_s}(D)$.

Let $f \in h_{X_s}(D)$ and $f^+ \in X_s(T)$ be its boundary value function. Suppose that $f^+(t)=\sum_0^\infty c_n\left(f^+\right)u_n^+(t), \quad -\pi\le t<\pi$, is the corresponding decomposition of this function on the basis $\{u_n{}^+(t)\}$ in $X_s(T)$. Consider the polynomials

$$f_N(r,t)=\sum_{n=0}^{N} c_n\left(f^+\right)u_n(r,t),\quad for\ re^{it}\in D$$

and

$$f_N{}^+(t)=\sum_{0}^{N} c_n\left(f^+\right)u_n(t),\quad for\ \ t\in(-\pi;\pi).$$

From the estimate (4), Corollary 2 and Theorem 3 we have

$$\|f-f_N\|_{h_X} = \sup_{0\le r<1}\|f_r-(f_N)_r\|_X \le const\left\|f^+-f_N{}^+\right\|_X \overset{N\to\infty}{\longrightarrow} 0.$$

The corollary is proved. ■

3. Dirichlet Problem with an Oblique Derivative

In this section we consider only a rearrangement-invariant Banach function space. The following theorem is proved in [4], [34], [35], [36].

Theorem 4. . *Let $X(T)$ be a rearrangement-invariant space. Then:*
i) *The system $\{e^{int}\}_{n\in Z}$ forms a basis in $X_b(T)$ $\Leftrightarrow$ $0<\alpha_X;\beta_X<1$.*
ii) *If $0<\alpha_X;\beta_X<1$, then $\{e^{int}\}_{n\in Z_+}$ forms a basis in $H^+_{X_b}(T)$.*

Using Corollary 4, we have $u\in h_{X_s}(D)$, if $\frac{\partial u}{\partial r}\in h_{X_s}(D)$. Corollary 5 implies the following

Corollary 6. *Let $X(T)$ be a rearrangement-invariant space with Boyd indices $0<\alpha_X;\beta_X<1$. Then:*
i) *The system $\{z^n\}_{n\in Z}$ forms a basis in $h_{X_s}(D)$.*
ii) *The system $\{z^n\}_{n\in Z+}$ forms a basis in $H^+_{X_s}(D)$.*

Consider the following boundary value problem with oblique derivatives

$$\Delta_{r,\phi}u=0,\quad u\in h^{(1)}_{X_s}(D), \tag{6}$$

$$\begin{aligned}\cos\phi\gamma\left(\frac{\partial u}{\partial r}\right)+sin\phi\gamma\left(\frac{\partial u}{\partial \phi}\right)\equiv\left(\cos\phi\,\frac{\partial u}{\partial r}+sin\phi\,\frac{\partial u}{\partial \phi}\right)\Bigg|_{r=1}=\\ =f(\phi)\in X_s(T),\ for\ \phi\in[-\pi;\pi),\end{aligned} \tag{7}$$

where $\Delta_{r,\phi}u=\frac{\partial^2 u}{\partial r^2}+\frac{1}{r}\frac{\partial u}{\partial r}+\frac{1}{r^2}\frac{\partial^2 u}{\partial \phi^2}$ is the Laplace operator in polar coordinates.

Let us establish the Noetherness of the problem (6), (7) and calculate its index. Let's prove the following theorem.

Theorem 5. *Let $X(T)$ be a rearrangement-invariant space with Boyd indices $0<\alpha_X;\beta_X<1$ and let Property (β) hold. Then the problem (6), (7) is a Noetherian and its index equals $\chi=-2$.*

Proof. First, it should be noted that every solution of this problem is also a solution of the following one

$$\Delta_{r,\phi}u=0,\ u\in h^{(1)}_p(D), \tag{8}$$

$$\begin{aligned}\gamma\left(\cos\phi\,\frac{\partial u}{\partial r}+sin\phi\,\frac{\partial u}{\partial \phi}\right)\equiv\left(\cos\phi\,\frac{\partial u}{\partial r}+sin\phi\,\frac{\partial u}{\partial \phi}\right)\Bigg|_{r=1}=\\ =f(\phi)\in L_p(T),\ for\ \phi\in[-\pi;\pi),\end{aligned} \tag{9}$$

where $1 < p < \frac{1}{\beta_X}$ is some number. In the case of $h_p^{(1)}$ and $L_p(T)$, it was proved that if $f = 0$, then $u = const$ (see, [6]). Thus, we have $dim Ker\gamma = 1$, for (8), (9). Taking into account that every solution of (6), (7) is also a solution of (8), (9), we conclude that $dim Ker\gamma = 1$, also holds.

Let $u \in h_{X_s}^{(1)}(D)$ be a solution of the problem (6), (7).

It is evident that $cos\ \phi\ \gamma(\ \frac{\partial u}{\partial r}) + sin\ \phi\ \gamma(\frac{\partial u}{\partial \phi})\ ) \in X_s(T)$. Let

$$\begin{aligned} u(r,\phi) &= a_0 + \sum_{n=1}^{\infty} \left(a_n \cos n\phi + b_n \sin n\phi\right) r^n, \\ f(\phi) &= a_0' + \sum_{n=1}^{\infty} \left(a_n' \cos n\phi + b_n' \sin n\phi\right), \end{aligned} \tag{10}$$

be decompositions of these functions with respect to the bases

$$\left\{\frac{1}{2}, r^n \cos\ n\ \phi, r^n \sin\ n\ \phi\right\}_{n\in N} \quad and \quad \left\{\frac{1}{2}, cos\ n\ \phi, sin\ n\ \phi\right\}_{n\in N},$$

in $h_{X_s}(D)$ and $X_s(T)$, respectively. Then, by Corollaries 4-5 and (6), (7), we formally have

$$\begin{aligned} \frac{\partial u(r,\phi)}{\partial r} &= \sum_{n=1}^{\infty} n r^{n-1} \left(a_n \cos n\phi + b_n \sin n\phi\right), \\ \frac{\partial u(r,\phi)}{\partial \phi} &= \sum_{n=1}^{\infty} n r^{n} \left(-a_n \sin n\phi + b_n \cos n\phi\right), \end{aligned} \tag{11}$$

and

$$\begin{aligned} \lim_{r\to 1} \frac{\partial u(r,\phi)}{\partial r} &= \sum_{n=1}^{\infty} n \left(a_n \cos n\phi + b_n \sin n\phi\right), \\ \lim_{r\to 1} \frac{\partial u(r,\phi)}{\partial \phi} &= \sum_{n=1}^{\infty} n \left(-a_n \sin n\phi + b_n \cos n\phi\right), \end{aligned}$$

$$\begin{aligned} f &= \left(\cos\phi \frac{\partial u}{\partial r} + \sin\phi \frac{\partial u}{\partial \phi}\right)\Bigg|_T = \lim_{r\to 1^-} \left(\cos\phi \frac{\partial u(r,\phi)}{\partial r} + \sin\phi \frac{\partial u(r,\phi)}{\partial \phi}\right) \\ &= \sum_{n=1}^{\infty} n \left(a_n \cos\phi \cos n\phi + b_n \cos\phi \sin n\phi + b_n \sin\phi \cos n\phi - a_n \sin\phi \sin n\phi\right) \\ &= \sum_{n=1}^{\infty} n \left(a_n \cos(n+1)\phi + b_n \sin(n+1)\phi\right). \end{aligned}$$

Therefore, if $u \in h^{(1)}_{X_s}(D)$ is a solution of the problem (6), (7), then the following relations hold

$$na_n = a'_{n+1}, \qquad nb_n = b'_{n+1} \ , \quad \forall n \in N$$

which implies

$$a_1 = a'_2, \quad b_1 = b'_2. \tag{12}$$

Assume $f = a_0 + a_1 \cos\phi + b_1 \sin\phi$. It is evident that if $f \neq 0, (i.e. \ |a_0| + |a_1| + |b_1| \neq 0)$, then the problem (6), (7) is unsolvable. Therefore, $codim\gamma \geq 3$. Let us prove that $codim\gamma = 3$. For this, we propose that $\forall f \in X_s(T) \backslash \{h : h = a_0 + a_1 \cos\phi + b_1 \sin\phi \ , a_i \in R, \ i = 0, 1, 2\}$, the problem (6), (7) has a solution $u \in h^{(1)}_{X_s}(D)$.

Instead of (10) we will use the following exponential decompositions

$$u(r;\phi) = \sum_{-\infty}^{\infty} A_n r^{|n|} e^{in\phi}, \qquad f(\phi) = \sum_{-\infty}^{\infty} A'_n \ e^{in\phi},$$

where

$$A_n = \begin{cases} \frac{1}{2}(a_n - ib_n), & n > 0 \\ a_0, & n = 0 \\ \frac{1}{2}\left(a_{|n|} + ib_{|n|}\right), & n < 0 \end{cases}$$

Therefore

$$c_n(f) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)e^{-int}dt = \begin{cases} \frac{1}{2}(a'_n - ib'_n), & n > 0 \\ a_0 & n = 0 \\ \frac{1}{2}\left(a'_{|n|} + ib'_{|n|}\right), & n < 0 \end{cases}$$

Thus we have

$$\frac{\partial u}{\partial r} = \sum_{n\neq 0} |n| A_n r^{|n|-1} e^{in\phi} \qquad and \qquad \frac{\partial u}{\partial \phi} = \sum_{n\neq 0} in A_n r^{|n|} e^{in\phi}. \tag{13}$$

From (10) it follows that

$$c_{-1}(f) = c_0(f) = c_1(f) = 0, \tag{14}$$

and

$$u(r;\phi)=A_0+\sum_{-\infty}^{-1}\frac{c_{n-1}(f)}{|n|}r^{|n|}e^{in\phi}+\sum_{1}^{\infty}\frac{c_{n+1}(f)}{n}r^{n}e^{in\phi}.$$

The relations (13) and (14) imply

$$\frac{\partial u(r;\phi)}{\partial r}=\sum_{-\infty}^{-1}c_{n-1}(f)r^{|n|-1}e^{in\phi}+\sum_{1}^{\infty}c_{n+1}(f)r^{n-1}e^{in\phi}. \tag{15}$$

Let us consider the following functions

$$\begin{aligned} u_1(r;\phi) &= \sum_{-\infty}^{-1}c_{n-1}(f)r^{|n|-1}e^{in\phi}, \\ u_2(r,\phi) &= \sum_{1}^{\infty}c_{n+1}(f)r^{n-1}e^{in\phi}. \end{aligned} \tag{16}$$

We assert that $u_i\in h_X,\ for\ \ i=1,2.$ It is sufficient to prove that $u_2\in h_X$ (then instead of $u_1(r,\phi)$ we can consider $\overline{u_1(r,\phi)}$, which has a form of $u_2(r,\phi)$). It is clear that $\Delta_{r,\phi}u_2(r,\phi)=0,\ \ for\ \forall(r,\phi)\in D$. Represent $u_2(r,\phi)$ in the form

$$u_2(r,\phi)=e^{-i\phi}r^{-2}V_2(r,\phi),$$

where $V_2(r,\phi)=\sum_2^\infty c_n(f)r^ne^{in\phi}$. It is obvious that $u_2(r,\phi)\to c_2(f)e^{i\phi}\ \ as\ \ r\to 0^+$, uniformly with respect to ϕ. Therefore for every $\delta\epsilon(0,1)$, the condition $\sup_{\delta<r<1}\|u_2(r,.)\|_{X(T)}<\infty$, holds if and only if $\sup_{\delta<r<1}\|V_2(r,.)\|_{X(T)}<\infty$, holds.

Let $g(\phi)=\sum_2^\infty c_n(f)e^{in\phi},\phi\in(-\pi,\pi)$. Taking into account the fact that the system$\{e^{int}\}_{n\in Z,}$ forms a basis in $X_s(T)$, and $f\in X_s(T)$, we have $g\in X(T)$. Then, by Theorem 1 and the estimate (4), we obtain

$$V_2(r,\phi)=\frac{1}{2\pi}\int_{-\pi}^{\pi}P_r(\phi-\theta)g(\theta)d\theta\Longrightarrow\ \ \|V_2(r,\phi)\|_{X(T)}\le c\,\|g\|_{X(T)}\,,$$

where $c>0$ is independent of g, which implies $V_2\in h_{X(T)}$. Consequently, if $u_2\in h_X$, then $\frac{\partial u}{\partial r}\in h_X$.

For $\frac{\partial u}{\partial\phi}$ we have

$$\frac{\partial u}{\partial\phi}\ =\ -i\sum_{-\infty}^{-1}c_{n-1}(f)r^{|n|}e^{in\phi}+\sum_{1}^{\infty}c_{n+1}(f)r^{n}e^{in\phi}=$$

$$
\begin{aligned}
&= -ir^{-1}e^{i\phi}\sum_{-\infty}^{-2}c_n(f)r^{|n|}e^{in\phi} + r^{-1}e^{-i\phi}\sum_{2}^{\infty}c_n(f)r^n e^{in\phi} = \\
&= -ir^{-1}e^{i\phi}V_1(r,\phi) + r^{-1}e^{-i\phi}V_2(r,\phi) = \\
&= -iru_1(r,\phi) + ru_2(r,\phi).
\end{aligned}
$$

Consequently, $\frac{\partial u}{\partial \phi} \in h_{X_s}$, which implies $u \in h_{X_s}^{(1)}$.
The theorem is proved.■

Note that the results of this work are applicable to Lebesgue spaces $L_p(\Omega)$, grand-Lebesgue spaces $L_{p)}(\Omega)$, Marcinkiewicz spaces, the weak-type L_p^w spaces, Orlicz spaces, Lorentz spaces and etc.

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