Journal of Contemporary Applied Mathematics
V. 15, No 1, 2025, January
ISSN 2222-5498, E-ISSN 3006-3183
https://doi.org/10.62476/jcam.151.9

Vector-valued Grand Hardy Classes

B.T. Bilalov*, S.R. Sadigova, Y. Sezer

Abstract. It is considered the vector-valued grand Lebesgue space $L_{p)}(X) \equiv L_{p)}(J; X)$, $1 < p < \infty$, and the concept of a t-basis, generated by some bilinear map (where $J = [-\pi, \pi))$, is introduced. It is proved that the exponential system $\mathscr{E} \equiv \left\{e^{int}\right\}_{n \in Z}$ forms a t-basis for $N_{p)}(X)$, when X is a UMD space, where $N_{p)}(X)$ is the closure of X-valued infinitely differentiable functions in $L_{p)}(X)$. The concept of the t-Riesz property of the system $\mathscr{E}$ in $N_{p)}(X)$ is defined. It is established that this system has the t-Riesz property, when X is a UMD space. Using these facts, the X-valued grand Hardy classes ${}_{m}H^{\pm}_{p)}(X)$ of X-valued analytic functions are introduced, and some of their properties are proved. The obtained results are applied to establish the t-basicity of the perturbed exponential system in $N_{p)}(X)$.

Key Words and Phrases: X-valued grand Hardy classes, t-basicity, t-Riesz property

2010 Mathematics Subject Classifications: 30E25, 41A58, 42B30

---

1. Introduction

The theory of classical Hardy spaces $H^{\pm}_{p}$ of analytic functions is sufficiently well- developed and it is illuminated in many excellent monographs (see, e.g. [1, 2, 3, 4, 5]). This theory has various applications in different branches of mathematics, such as harmonic analysis, differential equations, approximation theory, boundary value problems for analytic functions and others. Hardy classes are used to establish the basis properties of some perturbed trigonometric systems in various functional spaces (see, e.g. the works [6, 7, 8, 9, 10, 11, 12]). The abstract generalizations of these classes also have a long and deep history and the works of various mathematicians (see, e.g. the works [13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27, 28]) are devoted to them. It is also worth noting that, due to the applications in concrete problems of mechanics, mathematical physics and pure mathematics, interest in the so-called nonstandard function

---

*Corresponding author.

spaces has significantly increased. More information can be found, for example, in the monographs [29, 30, 31, 32, 33, 34] and their references. This class of function spaces includes the Lebesgue space with variable summability index, Morrey spaces, grand-Lebesgue spaces, Orlicz spaces, Marchinkievicz spaces and others. Taking this into account, we intend to consider the X-valued grand Lebesgue generalization of Hardy spaces.

The vector-valued grand Lebesgue space $L_{p)}(X) \equiv L_{p)}(J;X)$, $1 < p < \infty$, is considered and the concept of a t-basis, generated by some bilinear map (where $J = [-\pi, \pi)$) is introduced. It is proved that the exponential system $\mathscr{E} \equiv \{e^{int}\}_{n\in Z}$ forms a t-basis for $N_{p)}(X)$, when X is a UMD space, where $N_{p)}(X)$ is a closure of X-valued infinitely differentiable functions in $L_{p)}(X)$. The concept of the t-Riesz property of the system $\mathscr{E}$ in $N_{p)}(X)$ is defined. It is established that this system has the t-Riesz property, when X is a UMD space. Using these facts, the X-valued grand Hardy classes ${}_mH^{\pm}_{p)}(X)$ of X-valued analytic functions are introduces and some of their properties are proved. The obtained results are applied to establish the t-basicity of the perturbed exponential system in $N_{p)}(X)$.

2. Notations and auxiliary facts

2.1. Standard notations

N are positive integers; Z is a set of integers; $Z_+ = \{0\} \bigcup N$; R are real numbers; C are complex numbers; $L[M]$ is a linear span of M; $\overline{M}$ is a closure of M; χ_M is a characteristic function of M; B-space is a Banach space; $\|\cdot\|_X$ is a norm in X; X^* is a dual space of X; $[X;Y]$ is a B-space of bounded (linear) operators from X to Y; $[X] = [X;X]$; $\omega = \{z \in C : |z| < 1\}$; $\gamma = \partial\omega = \{z \in C : |z| = 1\}$; $\omega^c = C \backslash \overline{\omega}$; $\mathscr{B}$ is the set of all B-spaces; $KerT$ is a kernel of T; $R(T)$ is an image of T; $(\overline{\cdot})$ is a complex conjugation; $\delta_{i,j}$ is the Kronecker symbol; $p' : \frac{1}{p} + \frac{1}{p'} = 1$ is a conjugate number to $p \in [1, +\infty)$. $cardM$ is a cardinality of M; $\text{æ}_0 = card\,N$. $c > 0$ denotes a constant, may be different in different places.

2.2. The concepts of t-span, t-completeness, t-biorthogonality, t-basis

Let $X;Y;Z \in \mathscr{B}$ and $t : X \times Y \to Z$ be some bilinear map, which satisfies the condition

$$\exists \delta > 0 : \delta \|x\|_X \|y\|_Y \leq \|t(x;y)\|_Z \leq \delta^{-1} \|x\|_X \|y\|_Y , \forall (x;y) \in X \times Y. \quad (1)$$

Such bilinear map we will call t -map. In this case $t(x;y)$ we denote briefly as $xy =: t(x;y)$. Let $M \subset Y$ be some set. t-span of Mwe will denote as $L_t[M]$

and define by the relation

$$L_t[M] = \left\{ z \in Z : \exists \{(x_k; y_k)\}_1^{n_0} \subset X \times M \Rightarrow z = \sum_{k=1}^{n_0} x_k y_k \right\}.$$

Based on this notion we define the following concepts.

The system $\{y_k\}_{k\in N} \subset Y$ is t -complete in Z, if $\overline{L_t\left[\{y_k\}_{k\in N}\right]} = Z$ (the closure is taken in Z).

The system of operators $\{T_n\}_{n\in N} \subset [Z; X]$ we call t-biorthogonal to the system $\{y_n\}_{n\in N} \subset Y$, if $T_n(xy_k) = \delta_{nk}x$, $\forall x \in X$ & $\forall k;n \in N$.

Triple $\{X; Y; Z\} \subset \mathscr{B}$ we will call t_Y-invariant, if $\{(x_n; y_n)\} \subset X \times Y$: $\sum_n x_n y_n = 0 \Rightarrow \sum_n \vartheta(y_n) x_n = 0$, $\forall \vartheta \in Y^*$.

Triple $\{X; Y; Z\} \subset \mathscr{B}$ we will call t-dense, if finite t-linear combination of form $\sum_k x_k y_k$ with $(x_k; y_k) \in X \times Y$, $\forall k$, is dense in Z.

It is valid the following criterion for t-basicity.

Theorem 1. *Let the triple $\{X; Y; Z\} \subset \mathscr{B}$ is t_Y-invariant and t-dense. Then the system $\{y_n\}_{n\in N} \subset Y$ forms t-basis for Z if and only if the following assertions hold:*

(i) $\{y_n\}_{n\in N}$ is t-complete in Z;

(ii) $\{y_n\}_{n\in N}$ has t-biorthogonal system $\{T_n\}_{n\in N} \subset [Z; X]$;

(iii) The projectors $\{P_m\}_{m\in N}$:

$$P_m(z) = \sum_{n=1}^{m} T_n(z) y_n, \forall m \in N \& \forall z \in Z,$$

are uniformly bounded, i.e. $\sup\limits_m \|P_m\|_{[Z]} < +\infty$.

Proof. **Necessity.** Let $\{y_n\}_{n\in N} \subset Y$ forms t-basis for Z. Then it is evident that (i) holds. Consider the assertion (ii). From the uniqueness of decomposition

$$z = \sum_{n=1}^{\infty} x_n y_n, \forall z \in Z, \tag{2}$$

follows that $y_n \neq 0$, $\forall n \in N$, and for every fixed $n \in N$ the coefficient x_n is a linear map from Z to X, i.e. $x_n = T_n(z)$. Moreover, it is evident that it holds

$$T_n(xy_k) = \delta_{nk}x, \forall x \in X, \forall n; k \in N.$$

Let us prove that $T_n \in [Z;X]$, $\forall n \in N$. So, by $\hat{X}$ we denote the set of all sequences $\hat{x} = \{x_n\}_{n\in N} \subset X$ such that the series $\sum_{n=1}^{\infty} x_n y_n$ converges in Z. Define the norm $\|\cdot\|_{\hat{X}}$ in $\hat{X}$ by the expression

$$\|\hat{x}\|_{\hat{X}} = \sup_n \left\| \sum_{k=1}^{n} x_k y_k \right\|_Z , \forall \hat{x} = \{x_n\}_{n\in N} \in \hat{X}.$$

Completely analogously to the classical basis case it is proved that $\left(\hat{X}; \|\cdot\|_{\hat{X}}\right)$ is B- space regarding wise-coordinate linear operations. It is obvious that to each element $\hat{x} = \{x_n\}_{n\in N} \in \hat{X}$ the decomposition (2) corresponds a unique element $z \in Z$ and this correspondence denote by A, i.e. $z = A\hat{x}$. It is evident that A is a linear operator, moreover $KerA = \{0\}$ and $R(A) = Z$. Moreover

$$\|A\hat{x}\|_Z = \|z\|_Z = \left\| \sum_{n=1}^{\infty} x_n y_n \right\|_Z \leq \sup_m \left\| \sum_{n=1}^{m} x_n y_n \right\|_Z = \|\hat{x}\|_{\hat{X}} ,$$

and in result from the Banach' s theorem we obtain that the operator $A \in \left[\hat{X}; Z\right]$ is an isomorphism. From other point of view using the property (1) of t-map we have

$$\|T_n(z)\|_X = \frac{\|T_n(z)\|_X \|y_n\|_Y}{\|y_n\|_Y} \leq \frac{\|T_n(z)\, y_n\|_Z}{\delta \|y_n\|_Y} = \frac{\left\| \sum_{k=1}^{n} T_k(z)\, y_k - \sum_{k=1}^{n-1} T_k(z)\, y_k \right\|_Z}{\delta \|y_n\|_Y} \leq$$

$$\leq \frac{2\sup_m \left\|\sum_{k=1}^{m} T_k(z)\, y_k\right\|_Z}{\delta \|y_n\|_Y} = \frac{2\left\|\{T_n(z)\}_{n\in N}\right\|_{\hat{X}}}{\delta \|y_n\|_Y} =$$

$$= \frac{2\left\|A^{-1}z\right\|_{\hat{X}}}{\delta \|y_n\|_Y} \leq \frac{2\left\|A^{-1}\right\|}{\delta \|y_n\|_Y} \|z\|_Z , \ \forall z \in Z.$$

Consequently, $T_n \in [Z;X]$, $\forall n \in N$, and it holds

$$\|T_n\| \leq \frac{2\left\|A^{-1}\right\|}{\delta \|y_n\|_Y}.$$

In result we obtain the validity of the following inequality

$$1 \leq \|T_n\| \, \|y_n\|_Y \leq \frac{2}{\delta} \left\|A^{-1}\right\| , \forall n \in N. \tag{3}$$

The assertion (ii) is proved. Assertion (iii) follows immediately from the classical Banach-Steinhaus theorem.

Sufficient part of the theorem is proved completely analogously to the classical case.

Theorem is proved.

Consider the particular case, when Z is some Banach tensor product of spaces X; $Y \in \mathscr{B}$ and we will denote it by $Z = X\overline{\otimes}Y$. Along with by $X \otimes Y$ we denote the linear tensor product of spaces X&Y. The bilinear map $t : X \times Y \to Z$ we define as $t(x;y) = x \otimes y$, where $x \otimes y$ denotes the tensor product (an elementary tensor) of elements $x \in X$&$y \in Y$. It is evident that $X\overline{\otimes}Y$is t_Y- invariant and t-dense . Then according to the Theorem 1 we have the following

Corollary 1. *Let $X;Y \in \mathscr{B}$ and $Z = X\overline{\otimes}Y$. Then the system $\{y_n\}_{n\in N} \subset Y$ forms t-basis for Zif and only if the assertions (i)-(iii) of Theorem 1 hold.*

For completeness and simplicity of the subsequent presentations we need some facts about vector-valued Lebesgue space (Bochner space) $L_p(X) =: L_p(J;X)$. Firstly accept the following agreement. We identify the segment $J = [-\pi, \pi)$ and the unit circle γ by the mapping $e^{it} : J \leftrightarrow \gamma$. This allows us to identify the Bochner spaces $L_p(J;X)$ & $L_p(\gamma;X)$, defining on J and γ, respectively, and identify the function $f(t)$, $t \in J$, defining on J, with the function $f(\tau) = f\left(e^{it}\right) =: f(t)$, $\tau = e^{it} \in \gamma$. Let us accept $f_r(t) = f\left(re^{it}\right)$, $\forall r \geq 0$ & $t \in J$.

So, let $(S;\mathscr{A};\mu)$ be a measure space. Denote by $L_p(S;X)$ the Bochner space, generated by measure space $(S;\mathscr{A};\mu)$ and with norm

$$\|f\|_{L_p(S;X)} = \left(\int_S \|f\|_X^p \, d\mu\right)^{1/p} .$$

Recall the definition of UMD (Unconditional Martingale Difference) space.

Definition 1. *A B-space X is said to have the property of UMD, if for all $p \in (1,\infty)$ there exists a finite constant $\beta \geq 0$(depending on p and X) such that the following holds: whenever $(S;\mathscr{A};\mu)$ is a $\sigma-$finite measure space, $\{\mathscr{F}_n\}_{n=0}^N$ is a $\sigma-$finite filtration and $\{f_n\}_{n=0}^N$ is a finite martingale in $L_p(S;X)$, then for all scalar $|\varepsilon_n| = 1$, $n = \overline{1,N}$, we have*

$$\left\|\sum_{n=1}^N \varepsilon_n df_n\right\|_{L_p(S;X)} \leq \beta \left\|\sum_{n=1}^N df_n\right\|_{L_p(S;X)} ,$$

where $df_n = f_n - f_{n-1}$ is a martingale difference.

If this condition holds, then $X$is said to be a UMD space. In the sequel, the set of all UMD spaces we will denote by the same letter UMD.

Concerning relatively notions and more information, one can see f.e. the monography [21].

Let $\mathscr{E} = \{e_n(\cdot)\}_{n\in Z}$ is exponential system, where $e_n(t) = e^{int}$, $t \in J$. Recall that an X- valued trigonometric polynomial is a function $P_n : \gamma \to X$ of the form

$$P_n(\tau) = P_n(t) = \sum_{k=-n}^{n} a_k \tau^k = \sum_{k=-n}^{n} a_k e_k(t) \ , \ \ t \in J, \tag{4}$$

with coefficients $a_k \in X, \ k = \overline{-n, n}$. The set of all X-valued polynomials denote by $\mathscr{P}(X)$. It is valid the following (see, e.g. [21])

Proposition 1. *Let $X \in \mathscr{B} \,\&\, p \in [1, \infty)$. Then $\mathscr{P}(X)$ is dense in $L_p(X)$.*

Let $P \in \mathscr{P}(X)$, i.e. $P(\cdot)$ has the form (a finitely non-zero sum)

$$P(t) = \sum_{k\in Z} a_k e_k(t) \ .$$

Define the multiplier operator $\mathscr{M}$ by expression

$$\mathscr{M}(P) = \tilde{P} = -i \sum_{k\in Z} signk\, a_k e_k(\cdot) \ ,$$

where

$$signk = \begin{cases} 1\,, \ k \geq 1\,, \\ 0\,, \ k = 0\,, \\ -1\,, \ k \leq -1\,. \end{cases}$$

Denote by $L_p^0(X)$ the following subspace of $L_p(X)$:

$$L_p^0(X) = \left\{ f \in L_p(X) : \int_{-\pi}^{\pi} f(t)\, dt = 0 \right\}.$$

By Hwe denote the X- valued Hilbert transform on R:

$$Hf(x) = \frac{1}{\pi} \int_R \frac{f(y)}{y - x} dy\,, \ \ x \in R\,,$$

which is understood in the singular sense. It is known the following UMD property characterization of UMD space.

Theorem 2 (Burkholder-Baurgain)**.** *Let $X \in \mathscr{B} \& p \in (1, \infty)$. The following assertions are equivalent:*

(i) $X \in UMD$;

(ii) $H \in [L_p(R;X)]$.

In obtaining main results we will use the following very strongly result.

Proposition 2. *Let* $X \in \mathscr{B} \& p \in (1,\infty)$. *If* $H \in [L_p(R;X)]$, *then* $\mathscr{M} \in \left[L_p^0(X)\right]$ & $\mathscr{M} \in [L_p(X)]$.

For function $f \in L_1(X)$ by $\hat{f}_k$, $k \in Z$,we denote its t-Fourier coefficients

$$\hat{f}_k = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)\, e^{-ikt} dt,\ k \in Z\,.$$

It is evident that $\hat{f}_k \in X,\ \forall k \in Z$. Denote also by P_n the n-th order projection

$$P_n f = \sum_{|k|\le n} \hat{f}_k e_k\,,\ \ n \in Z\,.$$

We also will need the following

Proposition 3. *Let* $X \in \mathscr{B}$ & $p \in (1,\infty)$. *Then the following assertions are equivalent:*

(i) $\mathscr{M} \in [L_p(X)]$; *(ii) the projections* $\{P_n\}_{n\in Z}$ *are uniformly bounded in* $L_p(X)$; *(iii) for* $\forall f \in L_p(X)$ *one has* $P_n f \to f$, $n \to \infty$ *,in* $L_p(X)$.

We will also need the following X-valued Minkowski's inequality.

Proposition 4 ([21])**.** *Let* $(S_k;\mathscr{A}_k;\mu_k)$, $k = 1,2$, *be measure spaces and* $X \in \mathscr{B}$. *Then for all* $1 \le p_1 \le p_2 < \infty$, *it holds*

$$\|f\|_{L_{p_2}(S_2;\mu_2;L_{p_1}(S_1;X))} = \left(\int_{S_2}\left(\int_{S_1}\|f(x;y)\|_X^{p_1} d\mu_1(x)\right)^{p_2/p_1} d\mu_2(y)\right)^{1/p_2} \le$$

$$\le \left(\int_{S_1}\left(\int_{S_2}\|f(x;y)\|_X^{p_2} d\mu_2(y)\right)^{p_1/p_2} d\mu_1(x)\right)^{1/p_1} = \|f\|_{L_{p_1}(S_1;\mu_1;L_{p_2}(S_2;X))}\,. \tag{5}$$

2.3. Grand Bochner space

By $L_0(X)$ we will denote all measurable (in strong sense) X- valued functions on J (it is the same on γ). Grand Bochner space $L_{p)}(X)$, $1 < p < +\infty$, we define by the norm

$$\|f\|_{L_{p)}(X)} = \sup_{0<\varepsilon<p-1}\left(\varepsilon\int_J \|f(t)\|_X^{p-\varepsilon} dt\right)^{\frac{1}{p-\varepsilon}}.$$

Also by $L_{p)}(J)$ denote the usual grand Lebesgue space of scalar-valued functions on J with norm

$$\|f\|_{L_{p)}(J)} = \sup_{0<\varepsilon<p-1} \left(\varepsilon \int_J |f(t)|^{p-\varepsilon} dt \right)^{\frac{1}{p-\varepsilon}}.$$

It is known that $L_{p)}(J)$ is nonseparable B-space. From here immediately follows that $L_{p)}(X)$ is also nonseperable B-space. In fact, let $\{f_\alpha\}_{\alpha\in M} \subset L_{p)}(J)$ such that, $\exists\delta > 0 : \forall\alpha \neq \beta \Rightarrow \|f_\alpha - f_\beta\|_{L_{p)}(J)} \geq \delta$, where $cardM > æ_0$. Let $a \in X,\ a \neq 0$, be an arbitrary element and assume $F_\alpha(t) = f_\alpha(t)\, a,\ \forall\alpha \in M,\ t \in J$. It is evident that $\{F_\alpha\}_{\alpha\in M} \subset L_{p)}(X)$. We have

$$\|F_\alpha\|_{L_{p)}(X)} = \|a\|_X \|f_\alpha\|_{L_{p)}(J)},\ \forall\alpha \in M.$$

From here it directly follows that $\|F_\alpha - F_\beta\|_{L_{p)}(X)} \geq \delta \|a\|_X,\ \forall\alpha \neq \beta : \alpha; \beta \in M$, and in result $L_{p)}(X)$ is nonseparable.

Let us define the seperable subspace of $L_{p)}(X)$, in which the infinitely differentiable X -valued functions on $\bar{J}$ (we denote it by $C^\infty(X)$) is dense. Denote by T_δ the shift operator in $L_0(X)$:

$$(T_\delta f)(x) = \begin{cases} f(x+\delta), & x+\delta \in J, \\ 0, & x+\delta \notin J. \end{cases}$$

Assume

$$N_{p)}(X) = \left\{ f \in L_{p)}(X) : \|T_\delta f - f\|_{L_{p)}(X)} \to 0,\ \delta \to 0 \right\}.$$

It is not hard to see that $N_{p)}(X)$ is subspace (i.e. closure) of $L_{p)}(X)$. Let us show that $C^\infty(X)$ is dense in $N_{p)}(X)$. Consider the following averaging function

$$\omega_\varepsilon(t) = \begin{cases} c_\varepsilon \exp\left(-\frac{\varepsilon^2}{\varepsilon^2 - |t|^2}\right), & |t| < \varepsilon, \\ 0, & |t| \geq \varepsilon, \end{cases}$$

where $c_\varepsilon \int_R \omega_\varepsilon(t)\, dt = 1$. For function $f \in N_{p)}(X)$ set

$$f_\varepsilon(t) = (f * \omega_\varepsilon)(t) = (\omega_\varepsilon * f)(t) = \int_R \omega_\varepsilon(t-s) f(s)\, ds = \int_R \omega_\varepsilon(s) f(t-s)\, ds.$$

Here we assume that the function $f(\cdot)$ is extended to the entire real axis by zero. We have

$$\|f_\varepsilon(t) - f(t)\|_X = \left\| \int_R \omega_\varepsilon(s) [f(t-s) - f(t)]\, ds \right\|_X \leq \int_R \omega_\varepsilon(s) \|f(t-s) - f(t)\|_X\, ds.$$

Consequently, applying the Minkowski's inquality (5) (where we take $p_1 = 1 \& p_2 = p-\varepsilon$), we obtain

$$\left(\int_J \|f_\varepsilon(t)-f(t)\|_X^{p-\varepsilon}\,dt\right)^{1/p-\varepsilon} \le \left(\int_J\left(\int_R \omega_\varepsilon(s)\,\|f(t-s)-f(t)\|_X\,ds\right)^{p-\varepsilon}dt\right)^{1/p-\varepsilon} \le$$

$$\le \int_R\left(\int_J\left[\omega_\varepsilon(s)\,\|f(t-s)-f(t)\|_X\right]^{p-\varepsilon}dt\right)^{1/p-\varepsilon}ds =$$

$$= \int_R \omega_\varepsilon(s)\left(\int_J \|f(t-s)-f(t)\|_X^{p-\varepsilon}\,dt\right)^{1/p-\varepsilon}ds =$$

$$= \int_{|s|\le\varepsilon} \omega_\varepsilon(s)\left(\int_J \|f(t-s)-f(t)\|_X^{p-\varepsilon}\,dt\right)^{1/p-\varepsilon}ds.$$

From here it immediately follows

$$\|f_\varepsilon - f\|_{L_{p)}(J)} \le \int_{|s|\le\varepsilon} \omega_\varepsilon(s)\,\|f(\cdot-s)-f(\cdot)\|_{L_{p)}(J)}\,ds \Rightarrow \|f_\varepsilon - f\|_{L_{p)}(J)} \to 0,\ \varepsilon\to 0.$$

So, we have proved the following

Lemma 1. *The set $C^\infty(X)$ is dense in $N_{p)}(X)$.*

And now consider the following conjugate function operator (or periodic Hilbert transform)

$$\left(\tilde{H}f\right)(t) = \frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{f(\tau)}{tg\frac{\tau-t}{2}}d\tau = \frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{f(\tau-t)}{tg\frac{\tau}{2}}d\tau,\ \ t\in J,$$

where $f\in L_1(X)$ is X -valued function. It is known that (see e.g. the monographs [30, 31, 35]) $\tilde{H}\in\left[L_{p)}(J)\right]$, $1<p<+\infty$. Taking into account to the fact that

$$\mathscr{M}P = \tilde{H}P,\ \ \forall P\in\mathscr{P}(X),$$

(regarding this fact see, e.g. Bennet, Sharpley [35, p.162]), then from Proposition 2 it directly follows that $\tilde{H}\in[L_p(X)]$, $\forall p: 1<p<+\infty$, if $X\in UMD$. Using this fact completely analogously to the scalar grand Lebesgue case $L_{p)}(J)$, it is proved that $\tilde{H}\in\left[L_{p)}(X)\right]$, $1<p<+\infty$, if $X\in UMD$. So, it is valid the following

Proposition 5. *Let $X\in UMD$. Then $\tilde{H}\in\left[L_{p)}(X)\right]$, $\forall p: 1<p<+\infty$.*

Let's prove that $\tilde{H} \in \left[N_{p)}(X)\right]$. In fact, let $f \in N_{p)}(X)$ be an arbitrary function and $F(t) = \left(\tilde{H}f\right)(t)$, $t \in J$. We have

$$F(t-s) - F(t) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{f(\tau - t + s) - f(\tau - t)}{tg\frac{\tau}{2}} d\tau = \left(\tilde{H}g_s\right)(t),$$

where $g_s(\tau) = f(\tau + s) - f(\tau)$, $\tau \in J$. Consequently

$$\|F(t-s) - F(t)\|_{L_{p)}(X)} = \left\|\left(\tilde{H}g_s\right)\right\|_{L_{p)}(X)} \leq C\|g_s\|_{L_{p)}(X)} =$$

$$= C\|f(\cdot + s) - f(\cdot)\|_{L_{p)}(X)} \to 0, s \to 0, \ F \in N_{p)}(X).$$

In result we obtain the following

Proposition 6. *Let $X \in UMD$. Then $\tilde{H} \in \left[N_{p)}(X)\right]$, $\forall p : 1 < p < +\infty$.*

Let $N_{p)}(J) \otimes X$ be an algebraic tensor product of spaces $N_{p)}(J)$ and X. Denote by $N_{p)}(J) \underset{p)}{\bar{\otimes}} X$ the closure of $N_{p)}(J) \otimes X$ in $L_{p)}(X)$ (by the norm of $L_{p)}(X)$). It is evident that the algebraic tensor product $C^{\infty}\left(\bar{J}\right) \otimes X$ is dense in $N_{p)}(J) \otimes X$, and in result it also dense in $N_{p)}(J) \underset{p)}{\bar{\otimes}} X$. Since, every function $f \in C^{\infty}\left(\bar{J}\right) : \ f(-\pi) = f(\pi)$, can be approximated uniformly on $\bar{J}$ by trigonometric polynomials (it is the same by polynomials regarding the exponential system $\mathscr{E}$), then it is obvious that $\mathscr{P}(X)$ is dense in $C^{\infty}\left(\bar{J}\right) \otimes X$ (regarding the norm $\|\cdot\|_{L_{p)}(X)}$) and thus $\mathscr{P}(X)$ is dense in $N_{p)}(J) \underset{p}{\bar{\otimes}} X$. By the same reason (since $C^{\infty}(X)$ is dense in $N_{p)}(X)$) $\mathscr{P}(X)$ is also dense in $N_{p)}(X)$. From here it directly follows that $N_{p)}(X) = N_{p)}(J) \underset{p}{\bar{\otimes}} X$ (up to izometrically isometry) . Thus, it is valid

Statement 3. *The B-spaces $N_{p)}(X)$ and $N_{p)}(J) \underset{p}{\bar{\otimes}} X$ up to izometrically isometry coincide and $\mathscr{P}(X)$ is dense in $N_{p)}(X)$.*

2.4. t -basicity of $\mathscr{E}$ for $N_{p)}(X)$

In this section we will prove that the expontential system $\mathscr{E}$ forms t-basis for $N_{p)}(X)$, when $X \in UMD$. So, it is valid the following

Theorem 4. *Let $X \in UMD$. Then the exponential system $\mathscr{E}$ forms t-basis for $N_{p)}(X)$, $1 < p < +\infty$.*

Proof. Let's apply the t-basisness criterion (Corollary 1). From Statement 3 it directly follows the t-completeness of system $\mathscr{E}$ in $N_{p)}(X)$.

Let's prove the validity of assertion ii). Consider the operators

$$T_n f = \hat{f}_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)\, e^{-int} dt, \ \ \forall f \in N_{p)}(X), \forall n \in Z.$$

Take $\forall \varepsilon_0 \in (1, p-1)$. We have

$$\|T_n f\|_X \leq \frac{1}{2\pi}\int_{-\pi}^{\pi} \|f(t)\|_X \, dt \leq$$

$$\leq c\left(\varepsilon_0 \int_{-\pi}^{\pi} \|f(t)\|_X^{p-\varepsilon_0} dt\right)^{\frac{1}{p-\varepsilon_0}} \leq c\, \|f\|_{L_{p)}(X)}, \ \ \forall f \in N_{p)}(X).$$

Consequently, $\{T_n\}_{n\in Z} \subset \left[N_{p)}(X); X\right]$. Moreover, for arbitrary an elementary tensor $f(t) = e^{ikt} \otimes x, \ \ \forall x \in X \& \forall k \in Z$, we obtain

$$T_n\left(e^{ikt}x\right) = \delta_{nk} x, \quad \forall n; k \in Z.$$

Thus, the assertion ii) of Corollary 1 is also valid.

Let's check the validity of assertion iii) of Corollary 1. Consider the projectors

$$S_n f = \sum_{k=-n}^{n} e^{ikt} T_k(f) = \sum_{k=-n}^{n} e^{ikt} \hat{f}_k, \quad \forall n \in Z_+.$$

According to the classical results we have

$$S_n f = \frac{1}{2\pi}\int_{-\pi}^{\pi} D_n(\tau - t)\, f(\tau)\, d\tau, \quad \forall n \in Z_+,$$

where

$$D_n(t) = \frac{\sin\left(n+\frac{1}{2}\right)t}{2\sin\frac{t}{2}},$$

is a Dirichlet's kernel. Express $S_n f$ via the periodic Hilbert transform $\tilde{H}$:

$$(S_n f)(t) = \frac{1}{4\pi}\int_{-\pi}^{\pi} \frac{\sin(n+\frac{1}{2})(\tau-t)}{\sin\frac{\tau-t}{2}} f(\tau) d\tau$$

$$= \frac{1}{4\pi}\int_{-\pi}^{\pi} \frac{\sin n(\tau-t)\cos\frac{1}{2}(\tau-t) + \cos n(\tau-t)\sin\frac{1}{2}(\tau-t)}{\sin\frac{\tau-t}{2}} f(\tau) d\tau =$$

$$= \frac{1}{4\pi}\int_{-\pi}^{\pi}\frac{\sin n(\tau-t)}{\tan\frac{\tau-t}{2}}f(\tau)d\tau + \frac{1}{4\pi}\int_{-\pi}^{\pi}\cos n(\tau-t)f(\tau)d\tau.$$

Taking into account these relations from Proposition 6 we obtain

$$\|S_n f\|_{L_{p)}(X)} \le c\,\|f\|_{L_{p)}(X)}\,, \quad \forall n \in Z_+,$$

where the constant $c > 0$ independent of f and n. Theorem is proved.

3. Main results

In this section we firstly introduce the concept of t- Riesz Property of exponential system $\mathscr{E}$ in grand Bochner space $N_{p)}(X)$ and prove that $\mathscr{E}$ has t- Riesz Property in $N_{p)}(X)$. Then these results we apply to defining X-valued grand Hardy classes and to the investigation of its properties.

3.1. t-Riesz Property

Let $X \in UMD$ &$1 < p < +\infty$. Then by Theorem 4 the system $\mathscr{E}$ forms t-basis for $N_{p)}(X)$. Accept the following

Definition 2. *We say that the t- basis $\mathscr{E}$ of $N_{p)}(X)$ has t- Riesz property if* $\exists c > 0$*:*

$$\left\|\sum_{n=0}^{m_1} T_n(f)\,e^{int}\right\|_{L_{p)}(X)} \le c\,\|f\|_{L_{p)}(X)}\,; \quad \left\|\sum_{n=-m_2}^{-1} T_n(f)\,e^{int}\right\|_{L_{p)}(X)} \le c\,\|f\|_{L_{p)}(X)}\,,$$

$$\forall f \in N_{p)}(X)\,; \;\; \forall m_1; m_2 \in N,$$

where $\{T_n\}_{n\in Z} \subset \left[N_{p)}(X)\,; X\right]$ *is t-biorthogonal system to t- basis $\mathscr{E}$.*

It is valid the following

Theorem 5. *Let $X \in UMD$. Then the exponential system $\mathscr{E}$ has t-Riesz Property in $N_{p)}(X)\,,\; 1 < p < +\infty$.*

Proof. On the linear subspace $\mathscr{P}(X)$ of $N_{p)}(X)$, define the following operator R^+(we will call it as t-Riesz operator)

$$\left(R^+P\right)(\tau) = \sum_{n=0}^{m} a_n\tau^n, \quad \tau \in \gamma, \forall P \in \mathscr{P}(X) : P(\tau) = \sum_{n=-m}^{m} a_n\tau^n.$$

Let $\{T_n\}_{n\in Z} \subset \left[N_{p)}(X);X\right]$ be t- biorthogonal to $\mathscr{E}$ system and $\mathscr{M}$ is multiplier operator, defined in Section 2.2. It is not hard to see that

$$R^+P = \frac{1}{2}T_0(P) + \frac{1}{2}(P + i\mathscr{M}(P)). \tag{6}$$

We have

$$\|T_0(P)\|_{L_{p)}(X)} = \left\|\tfrac{1}{2\pi}\int_{-\pi}^{\pi} P\left(e^{it}\right)dt\right\|_{L_{p)}(X)} =$$

$$= c\sup_{0<\varepsilon<p-1}\left(\varepsilon\int_{-\pi}^{\pi}\left\|\int_{-\pi}^{\pi} P\left(e^{it}\right)dt\right\|_X^{p-\varepsilon}dx\right) \leq$$

$$\leq c\sup_{0<\varepsilon<p-1}\left(\varepsilon\int_{-\pi}^{\pi}\left\|P\left(e^{it}\right)\right\|_X^{p-\varepsilon}dt\right)^{1/p-\varepsilon} = c\|P\|_{L_{p)}(X)}.$$

Taking into account this inequality, from the relation (6) we obtain that $R^+ \in \left[L_{p)}(X)\right] \Leftrightarrow \mathscr{M} \in \left[L_{p)}(X)\right]$. Moreover, it is evident that $T_0 \in \left[N_{p)}(X)\right]$. Then again from (6) follows $R^+ \in \left[N_{p)}(X)\right] \Leftrightarrow \mathscr{M} \in \left[N_{p)}(X)\right]$. Take $\forall f \in N_{p)}(X)$ and let $S_m(f)$ be its m-th order partial sum

$$S_m(f)(\tau) = \sum_{n=-m}^{m} T_n(f)\tau^n, \quad \tau\in\gamma, \ m\in Z_+.$$

For m-th order polynomials $P(\cdot)$ of the form (4) we have

$$\left(R^+P\right)(\tau) = \tau^m\left[S_m\left(\tau^{-m}P(\tau)\right)\right](\tau), \quad \tau\in\gamma. \tag{7}$$

Since the system $\mathscr{E}$ forms t-basis for $N_{p)}(X)$, we have

$$\|S_m(f)\|_{L_{p)}(X)} \leq c\|f\|_{L_{p)}(X)}, \quad \forall m\in N; \ \forall f\in N_{p)}(X).$$

Then from (7) we obtain

$$\left\|R^+P\right\|_{L_{p)}(X)} \leq \left\|S_m\left(\tau^{-m}P(\tau)\right)\right\|_{L_{p)}(X)} \leq c\left\|\tau^{-m}P(\tau)\right\|_{L_{p)}(X)} \leq$$

$$\leq c\|P\|_{L_{p)}(X)}, \quad \forall P\in\mathscr{P}(X). \tag{8}$$

Taking into account the Statement 3 from inequality (8) we obtain that the operator R^+ extends continuosly to $N_{p)}(X)$, i.e. $R^+ \in \left[N_{p)}(X)\right]$. Therefore, for $\forall f \in N_{p)}(X)$ the series

$$R^+f = \sum_{n=0}^{\infty} T_n(f)\tau^n,$$

converges in $N_{p)}(X)$.

Completely analogously we prove that the operator

$$R^{-}f=\sum_{n=1}^{\infty}T_{-n}(f)\,\tau^{-n},$$

well defined for $\forall f\in N_{p)}(X)$ and $R^{-}\in\left[N_{p)}(X)\right]$.

Theorem is proved.

It is not hard to see that the operators R^{+} and R^{-} mutually disjoint, i.e. $R^{+}R^{-}=R^{-}R^{+}=0$, and moreover, it holds $I=R^{+}+R^{-}$ ($I\in\left[N_{P)}(X)\right]$ is an identity operator). Moreover, it is evident that for $\forall n\in Z$, the operators

$$R_{n}^{+}f=\sum_{k=n}^{\infty}\hat{f}_{k}\tau^{k}\,;\;R_{n}^{-}f=\sum_{k=-n+1}^{\infty}\hat{f}_{-k}\tau^{-k},\tag{9}$$

well defined for $\forall f\in N_{p)}(X)$ and $R_{n}^{\pm}\in\left[N_{p)}(X)\right]$. Consequently, $R_{0}^{\pm}=R^{\pm}$. Again for the above reason we have

$$R_{n}^{+}R_{n}^{-}=R_{n}^{-}R_{n}^{+}=0;I=R_{n}^{+}+R_{n}^{-},\forall n\in Z.$$

So, it is valid the following

Corollary 2. *Let $X\in UMD$. Then for t-Riesz operators $R_{n}^{\pm}$, defined by expressions* (9), *for $\forall n\in Z$, it is valid: i) $R_{n}^{\pm}\in\left[N_{p)}(X)\right]$, $\forall n\in Z$; ii) $(R_{n}^{\pm})^{2}=R_{n}^{\pm}$; iii) $R_{n}^{+}R_{n}^{-}=R_{n}^{-}R_{n}^{+}=0$; iv) $I=R_{n}^{+}+R_{n}^{-}$.*

3.2. Grand Cauchy-Bochner type integral

Based on the concept of t-Riesz operators (at the same time also projectors) in this subsection we will define X-valued grand Cauchy-Bochner type integrals .

Thus, denote by $P_{r}(\cdot)$ the following Poisson kernel for unit circle

$$P_{r}(t)=\frac{1-r^{2}}{1-2r\cos t+r^{2}},re^{it}\in\omega.$$

Let $X\in UMD$. Then by Theorem 4 the system $\mathscr{E}$ forms t-basis for $N_{p)}(X)$ and let$\{T_{n}\}_{n\in Z}\subset\left[N_{p)}(X);X\right]$ be corresponding t-biorthogonal system. Therefore, $\forall f\in N_{p)}(X)$ has expansion

$$f(t)=\sum_{k\in Z}\hat{f}_{k}e^{ikt},\tag{10}$$

where $\hat{f}_k = T_k(f)$, $k \in Z$. Denote $f_+ = R^+ f$:

$$f_+(t) = \sum_{k=0}^{\infty} \hat{f}_k e^{ikt}, \tag{11}$$

where R^+ be t-Riesz operator (Theorem 5). Taking into account to the obvious relation

$$\left(re^{it}\right)^k = \frac{1}{2\pi} \int_{-\pi}^{\pi} P_r(t-s)\, e^{iks} ds, \forall k \in Z_+, \forall re^{it} \in \omega,$$

from (11) we directly obtain (since $R^+ \in \left[N_{p)}(X)\right]$)

$$\sum_{k=0}^{\infty} \hat{f}_k \left(re^{it}\right)^k = \frac{1}{2\pi} \int_{-\pi}^{\pi} P_r(t-s)\, f_+(s)\, ds.$$

Assume

$$F(z) = \sum_{k=0}^{\infty} \hat{f}_k z^k, z = re^{it} \in \omega.$$

It is very known that $F(\cdot)$ is X-valued analytic in ω function (see, e.g. the monograph [22]). Thus, $F(\cdot)$ has Poisson-Bochner integral representation

$$F\left(re^{it}\right) = \frac{1}{2\pi} \int_{-\pi}^{\pi} P_r(t-s)\, f_+(s)\, ds. \tag{12}$$

Then according to the results of monograph [22] it is valid the following analogous of classical Fatou's theorem.

Theorem 6. *Let $X \in UMD$ and $R^+ \in \left[N_{p)}(X)\right]$, $1 < p < +\infty$, be Riesz operator. Then for $\forall f \in N_{p)}(X)$ the Poisson-Bochner integral (12) with $f_+ = R^+ f$ presents analytical function $F(\cdot)$ in ω and $F\left(re^{it}\right)$ converges strongly (i.e. by norm $\|\cdot\|_X$) nontangential to $f_+\left(e^{it}\right)$ under $\omega \ni z \to e^{it}$ at point e^{it} , for which*

$$\frac{1}{2s} \int_{t-s}^{t+s} f_+(\tau)\, dt \to f_+(t), s \to 0,$$

i.e. almost everywhere.

By the same reason as above using the relation

$$z^k = \frac{1}{2\pi i} \int_{\gamma} \frac{\tau^k d\tau}{\tau - z}, \forall z \in \omega, \forall k \in Z_+,$$

for function $F(\cdot)$ we obtain the following Cauchy-Bochner representation

$$F(z) = \frac{1}{2\pi i} \int_\gamma \frac{f_+(\tau)\, d\tau}{\tau - z}, z \in \omega, f_+ = R^+ f.$$

A nontangential limit $\lim\limits_{\omega \ni z \to \tau} F(z)$ at point $\tau \in \gamma$ denote by $F^+(\tau)$. Consequently, according to the Theorem 6 it holds $F^+(\tau) = f_+(\tau)$, a.e. $\tau \in \gamma$.

Analogously results we obtain regarding the function

$$f_-(t) = \left(R^- f\right)(t) = \sum_{n=1}^{\infty} \hat{f}_{-n} e^{-int},$$

where $R^-$is corresponding Riesz operator. Define the following X-valued analytic in ω^c function

$$\Phi(z) = -\sum_{n=1}^{\infty} \hat{f}_{-n} z^{-n}, z \in \omega^c.$$

It is evident that $\Phi(\infty) = 0$. Completely analogously to the previous case, using the formula

$$z^{-k} = -\frac{1}{2\pi i} \int_\gamma \frac{\tau^{-k}}{\tau - z} d\tau, \forall z \in \omega^c, \forall k \in N,$$

we establish that the function $\Phi(\cdot)$ has the following X-valued Cauchy-Bochner representation

$$\Phi(z) = -\frac{1}{2\pi i} \int_\gamma \frac{f_-(\tau)}{\tau - z} d\tau\,, \quad \forall z \in \omega^c\,.$$

Introduce to the consideration the following analytic in ω function

$$\tilde{\Phi}(z) = \Phi\left(\frac{1}{z}\right)\ , \quad z \in \omega.$$

We have

$$\tilde{\Phi}(z) = -\sum_{n=1}^{\infty} \hat{f}_{-n} z^n\ , \quad z \in \omega\,.$$

Consider the following series (informal for now)

$$\tilde{f}\left(e^{it}\right) = -\sum_{n=1}^{\infty} \hat{f}_{-n} e^{int}\,.$$

It is not hard to see that along with series (10) (for function $f(t)$ in $N_{p)}(X)$) the following series

$$f(-t) = \sum_{n \in Z} \hat{f}_n e^{-int} = \sum_{n \in Z} \hat{f}_{-n} e^{int}\,,$$

also converges in $L_{p)}(X)$. Then as previous case we establish that the function $\tilde{\Phi}(\cdot)$ has a non-tangential values $\tilde{\Phi}^+(e^{it})$ on γ inside of ω and it holds

$$\tilde{\Phi}^+\left(e^{it}\right) = \tilde{f}\left(e^{it}\right) = -\sum_{n=1}^{\infty} \hat{f}_{-n} e^{int}, \ \ a.e.\ t \in J.$$

From here we immediately obtain that the function $\Phi(\cdot)$ also has nontangential values $\Phi^-\left(e^{it}\right)$ on γ outside of ω (i.e. inside of) and $\Phi^-\left(e^{it}\right)$ has the representation in $N_{p)}(X)$:

$$\Phi^-\left(e^{it}\right) = -\sum_{n=1}^{\infty} \hat{f}_{-n} e^{-int}.$$

Moreover, it is obvious that it is true the following relations

$$\frac{1}{2\pi i}\int_{\gamma}\frac{\tau^k d\tau}{\tau - z} = \begin{cases} z^k,\ z \in \omega, \\ 0,\ z \in \omega^c, \end{cases} \quad \forall k \in Z_+,$$

$$\frac{1}{2\pi i}\int_{\gamma}\frac{\tau^{-n} d\tau}{\tau - z} = \begin{cases} -z^{-n},\ z \in \omega^c, \\ 0,\ z \in \omega, \end{cases} \quad \forall n \in N.$$

Taking into account these relations as previous cases we establish the validity of the following X-valued Cauchy-Bochner formulas

$$\frac{1}{2\pi i}\int_{\gamma}\frac{f_+(\tau)\,d\tau}{\tau - z} = \begin{cases} \sum_{n=0}^{\infty}\hat{f}_n z^n,\ \ z \in \omega, \\ 0,\ \ z \in \omega^c, \end{cases} \tag{13}$$

$$-\frac{1}{2\pi i}\int_{\gamma}\frac{f_-(\tau)\,d\tau}{\tau - z} = \begin{cases} 0,\ \ z \in \omega, \\ \sum_{n=1}^{\infty}\hat{f}_{-n} z^{-n},\ z \in \omega^c, \end{cases} \tag{14}$$

where $f \in N_{p)}(X) : f_{\pm} = R^{\pm} f$.

And now consider the following X-valued Cauchy-Bochner type integral

$$F(z) = \frac{1}{2\pi i}\int_{\gamma}\frac{f(\tau)\,d\tau}{\tau - z},\ \ z \notin \gamma. \tag{15}$$

Putting instead of f the expression $f = R_+ f + R_- f = f_+ + f_-$, and using the formulas (13); (14) we obtain

$$F(z) = \begin{cases} \sum_{n=0}^{\infty}\hat{f}_n z^n,\ z \in \omega, \\ -\sum_{n=1}^{\infty}\hat{f}_{-n} z^{-n},\ z \in \omega^c, \end{cases}$$

where

$$f(\tau) = \sum_{n \in Z} \hat{f}_n \tau^n \ , \ \tau \in \gamma \ ,$$

is an expansion of f on t-basis $\mathscr{E}$ in $N_{p)}(X)$. From here it directly follows that the function $F(\cdot)$ has a nontangential values $F^{\pm}(\cdot)$ a.e. on γ from inside and outside of ω, respectively, and for these values it is valid

$$\left.\begin{array}{l} F^{+}(\tau) = f_{+}(\tau), \ a.e. \ \tau \in \gamma, \\ \\ F^{-}(\tau) = -f_{-}(\tau), a.e. \ \tau \in \gamma. \end{array}\right\} \tag{16}$$

Consequently, we have

$$F^{+}(\tau) - F^{-}(\tau) = f(\tau) \ , a.e. \ \tau \in \gamma.$$

Thus, we have proved the following

Theorem 7. *Let $X \in UMD$. Then for $\forall f \in N_{p)}(X)$, $1 < p < +\infty$, the X-valued Cauchy- Bochner type integral* (15) *is X-valued analytic on $C \backslash \gamma$ function and it has non-tangential values $F^{\pm}(\cdot)$ a.e. on γ from inside* (" + ") *and outside* (“- “) *of ω, respectively, moreover these limits satisfy*

$$F^{\pm}(\tau) = \pm \left(R^{\pm} f\right)(\tau), \ a.e. \ \tau \in \gamma;$$

$$F^{+}(\tau) - F^{-}(\tau) = f(\tau), \ a.e. \ \tau \in \gamma,$$

where $R^{\pm}$ are corresponding t-Riesz operators.

3.3. Grand Hardy-Bochner classes ${}_nH^{\pm}_{p)}(X)$

Here using the results, obtained in the previous subsection, we will define X-valued grand Hardy spaces ${}_nH^{\pm}_{p)}(X)$ of X-valued analytic in ω (sign is "+") and ω^c (sign is "-") functions, respectively. So, let $X \in UMD$ and $R^{\pm}_n \in \left[N_{p)}(X)\right]$, $1 < p < +\infty$, $n \in Z$, be t-Riesz projectors, defined by formulas (9) . Assume

$${}_nN^{\pm}_p(X) = R\left(R^{\pm}_n\right) = R^{\pm}\left(\ N_p(X)\right) \ , \tag{17}$$

i.e. ${}_nN^{\pm}_{p)}(X)$ is an image of the operator $R^{\pm}_n$. It is evident that ${}_nN^{\pm}_{p)}(X)$ is subspace (i.e. closure) of $N_{p)}(X)$. It directly follows from the fact that $R^{\pm}_n$ is a continuous projector in $N_{p)}(X)$.

For simplicity the following presentation let's define some class of functions.

By $\mathscr{A}^{\pm}(X)$ we denote the set of all X-valued analytic in ω $(sign\ "+")$ and ω^c $(sign\ "-")$ functions, respectively. Also denote by $\mathscr{K}f$ the X-valued Cauchy-Bochner type integral

$$(\mathscr{K}f)(z) = \frac{1}{2\pi i}\int_{\gamma}\frac{f(\tau)}{\tau - z}d\tau, \ \ z \notin \gamma\,.$$

Definition 3. *Let $X \in UMD$ and $R_n^{\pm} \in \left[N_{p)}(X)\right]$, $1 < p < +\infty$, $n \in Z$, be t-Riesz projectors. Define the following grand Hardy – Bochner classes*

$${}_nH_{p)}^{\pm}(X) = \left\{\Phi^{\pm} \in A^{\pm}(X) : \exists f^{\pm} \in {}_nN_p^{\pm}(X) \Rightarrow \Phi^{\pm}(z) = \left(\mathscr{K}f^{\pm}\right)(z)\right\},$$

provided with the norm

$$\left\|\Phi^{\pm}\right\|_{{}_nH_{p)}^{\pm}(X)} = \left\|f^{\pm}\right\|_{L_{p)}}(X)\,.$$

Scalar valued case (i.e. when $X \equiv C$) these subspaces we will denote as $N_{p)}(J) = N_{p)}(J;C)$; ${}_nN_p^{\pm}(J) = {}_nN_p^{\pm}(J;C)$; ${}_nH_{p)}^{\pm} = {}_nH_{p)}^{\pm}(J;C)$.

Let's prove that this definition is correct. For this let us show that the representation

$$\Phi^{\pm}(z) = \frac{1}{2\pi i}\int_{\gamma}\frac{f^{\pm}(\tau)}{\tau - z}d\tau, \ \ f^{\pm} \in {}_nN_{p)}^{\pm}(X)\ ,$$

is unique. It is sufficient to prove this for the case ${}_0H_{p)}^{\pm}(X)$. So , let for some $f_0{}^{\pm} \in {}_0N_{p)}^{\pm}(X)$ it holds

$$\int_{\gamma}\frac{f^{+}(\tau)\,d\tau}{\tau - z} \equiv 0\,, \forall z \in \omega\,. \tag{18}$$

In the classical case (i.e. scalar valued function case) this definition is correct (see, e.g. the monographs [1, 2, 3, 4]) , i.e. if for some $g^{+} \in {}_0L_q^{+}(J)\,, 1 < q < +\infty$, it holds

$$\int_{\gamma}\frac{g^{+}(\tau)}{\tau - z}d\tau = 0\,, \forall z \in \omega\,,$$

then $g^{+}(\tau)$=0 , a.e. $\tau \in \gamma$, where ${}_0L_q^{+}(J)$ is subspace of ordinary Lebesgue space $L_q(J)$, defined according to the previous definition. It is not hard to see that for $\forall\varepsilon_0 \in (1, p-1)$ the continuous embedding $N_{p)}(J) \subset L_{p-\varepsilon_0}(J)$ is valid. From here also follows the continuous embedding ${}_nN_{p)}^{\pm}(J) \subset {}_nL_{p-\varepsilon_0}^{\pm}(J)$ and therefore it is evident that the Definition 3. is correct for the scalar case ${}_nN_{p)}^{\pm}(J)$, too.

Take $\forall\vartheta \in X^*$and let $\varepsilon_0 \in (1, p-1)$- arbitrary fixed number. It is evident that $\vartheta\left(f_0^+\right) = g_\vartheta^+ \in {}_0N_{p)}^+(J) \subset {}_0L_{p-\varepsilon_0}^+(J)$ and from (18) it direct follows

$$\int_\gamma \frac{g_\vartheta^+(\tau)}{\tau - z} d\tau = 0\,,\ \forall z \in \omega\,.$$

Consequently, $g_\vartheta^+(\tau) = 0$, a.e. $\tau \in \gamma$. Since $f^+ \in {}_0N_{p)}^+(X)$, then f^+ has in $N_{p)}(X)$ the expansion (by definition of ${}_0N_{p)}^+(X)$)

$$f^+(\tau) = \sum_{n=0}^{\infty} \hat{f}_n \tau^n.$$

From here we obtain that in $N_{p)}(J)$ it holds

$$0 = g_\vartheta^+(\tau) = \sum_{n=0}^{\infty} \vartheta\left(\hat{f}_n\right)\tau^n\,. \tag{19}$$

It is obvious that this series also converges in $L_{p-\varepsilon_0}(J)$. Since the system $\mathscr{E}$forms basis for $L_{p-\varepsilon_0}(J)$ (because $p-\varepsilon_0 > 1$), from (19) follows that $\vartheta\left(\hat{f}_n\right) = 0\,, \forall n \in Z_+$. From arbitrariness of $\vartheta \in X^*$ we obtain $\hat{f}_n\,, = 0\,, \forall n \in Z_+ \Rightarrow f^+ = 0\,$. In result we have proved the following

Lemma 2. *Let $X \in UMD$. Then the Definition 3 of grand Hardy- Bochner classes ${}_nH_{p)}^\pm(X)$, $1 < p < +\infty$, is correct.*

It is evident that the classes ${}_nH_{p)}^\pm(X)$, $1 < p < +\infty$, with norm $\|\cdot\|_{{}_nH_{p)}^\pm(X)}$ are B-spaces. For simplicity in the sequel we accept the notations $H_{p)}^+(X) = {}_0H_{p)}^+(X)$; $H_{p)}^-(X) = {}_{-1}H_{p)}^-(X)$. Also we denote by $\mathscr{P}_n^\pm(X)$ the set off all polynomials of order $\leq n \in N$, with X- valued coefficients of the form

$$P_n^\pm(z) = \sum_{k=0}^{n} a_k^\pm z^{\pm k}\,,\ a_0^- = 0\,,\ z \in C\backslash\{0\}\ ,\left\{a_k^\pm\right\} \subset X. \tag{20}$$

It is evident that if $X \in UMD$ & $n \in N$, then the systems $\left\{\tau^k : k = \overline{-n, \infty}\right\}$ and $\left\{\tau^k : k = \overline{-\infty, n}\right\}$ form t-basis for subspaces ${}_{-n}N_{p)}^+(X)$ and ${}_nN_{p)}^-(X)$, respectively, i.e.

$$\forall f \in {}_{-n}N_{p)}^+(X) \Rightarrow f(\tau) = \sum_{k=-n}^{\infty} \hat{f}_k \tau^k\,,\ \tau = \gamma\,,$$

$$\forall g \in {}_nN^-_{p)}(X) \Rightarrow g(\tau) = \sum_{k=-\infty}^{n} \hat{f}_k \tau^k \,, \; \tau = \gamma \,.$$

Taking into account the Theorem 7 from uniqueness of such expansions we obtain the validity of the following direct sums

$$_{-n}H^+_{p)}(X) = H^+_{p)}(X) \dot{+} \mathscr{P}^-_n(X) \; ; \; {}_nH^-_{p)}(X) = H^-_{p)}(X) \dot{+} \mathscr{P}^+_n(X) \,. \tag{21}$$

Consequently, it is valid the following

Statement 8. *Let $X \in UMD$ & $n \in N$. Then for grand Hardy-Bochner classes $_{\mp n}N^{\pm}_{p)}(X)$ it hold the direct sums* (21).

3.4. X- valued Sokhotski-Plemelj formulas

Along with X- valued Cauchy type integral consider the following X-valued Cauchy-Bochner singular integral

$$(Sf)\,(\tau) = \frac{1}{2\pi i}\int_\gamma \frac{f(\xi)\,d\xi}{\xi - \tau}, \, \tau = \gamma \,, \tag{22}$$

with density $f \in N_{p)}(X)$. Existence a.e. $\tau \in \gamma$ of singular integral Sf follows from Theorem 2 (Burkholder-Bourgain), (see, e.g. monograph [21; p.374]. Consider also Cauchy-Bochner type integral

$$F(z) = \frac{1}{2\pi i}\int_\gamma \frac{f(\xi)\,d\xi}{\xi - z}, \; z \notin \gamma \,, \tag{23}$$

with the same density $f(\cdot)$. In work [37] it is proved that if $X \in UMD \,\&\, X^*$ be separable and $f \in L_q(X)$,$1 < q < +\infty$, then for nontangential values $F^{\pm}(\cdot)$ of function $F(\cdot)$ on γ it is true the following X-valued Sokhotski-Plemelj formulas

$$F^{\pm}(\tau) = \pm\frac{1}{2} f(\tau) + (Sf)\,(\tau) \;, \text{a.e.}\tau = \gamma. \tag{24}$$

Taking into account to the continuous embedding $N_{p)}(X) \subset L_{p-\varepsilon}(X)\,, \forall \varepsilon \in (0, p-1)$, from here we obtain the validity of the formulas (24) in case $f \in N_{p)}(X)$, too. Also taking attention to the Theorem 7 and to the expressions of t-Riesz operators $R^{\pm}$ we arrive to the following result.

Theorem 9. *Let $X \in UMD \& X^*$ be separable. Then for $\forall f \in N_{p)}(X)\,, 1 < p < +\infty$, regarding the Cauchy-Bochner singular integral* (20) *and Cauchy-Bochner type integral* (21) *the following assertions hold:*

(i) it is true the X-valued Sokhotski-Plemelj formulas (24);

(ii) $(Sf)(\tau) = F^{\pm}(\tau) \mp \frac{1}{2} f(\tau) = \left[\pm\left(R^{\pm} \mp \frac{1}{2} I\right) f\right](\tau)$, *a.e.* $\tau \in \gamma$;

(iii) if $f(\tau) = \sum_{n=-\infty}^{+\infty} \hat{f}_n(\tau)$, *then* $(Sf)(\tau) = \frac{1}{2}\left(\hat{f}_0 + \sum_{n \neq 0} sign \hat{f}_n \tau^n\right)$;

(iv) $S \in \left[N_{p)}(X)\right]$.

As is known, $S \in \left[L_{q)}(X)\right], 1 < q < +\infty$. Moreover, note that for Bochner spaces$L_{p)}(S; X), 0 < p \leq \infty$,(where X be any B-space) the Marchinkiewicz interpolation theorem is also valid (see, e.g. the monograph [21, p.86], Theorem 2.2.3 (Marchinkiewicz)). Then completely analogously to the scalar case $L_{p)}(J)$ it is proved the validity of the following

Statement 10. *Let* $X \in UMD$. *Then* $S \in \left[L_{q)}(X)\right], \forall p : 1 < p < +\infty$.

3.5. X-valued grand Riesz theorem

In this subsection we establish the analogous of classical Riesz theorem for the grand Hardy-Bochner classes case. It is valid the following

Theorem 11. *Let* $X \in UMD$. *Then for* $\forall F \in H_{p)}^{+}(X)$, $1 < p < +\infty$, *the following relations hold:*

(i) $\|F_r(\cdot)\|_{L_{p)}(X)} \to \|F^{+}(\cdot)\|_{L_{p)}(X)}, r \to 1-0$;

(ii) $\|F_r(\cdot) - F^{+}(\cdot)\|_{L_{p)}(X)} \to 0, r \to 1-0$.

Proof. Let $\forall F \in H_{p)}, 1 < p < +\infty$. Then as we have proved the validity of the following Poisson- Bochner representation

$$F\left(re^{it}\right) = \frac{1}{2\pi}\int_{-\pi}^{\pi} P_r(s-t) F^{+}(s) ds, \forall re^{it} \in \omega,$$

where $P_r(\cdot)$ is a Poisson kernel. Let $\varepsilon \in (0, p-1)$ be an arbitrary number. Based on this representation completely analogously to the classical case it is proved that

$$\|F_r(\cdot)\|_{L_{p-\varepsilon}(X)} \leq \left\|F^{+}(\cdot)\right\|_{L_{p-\varepsilon}(X)}, \forall r \in (0,1).$$

From here it directly follows

$$\|F_r(\cdot)\|_{L_{p)}(X)} \leq \left\|F^{+}(\cdot)\right\|_{L_{p)}(X)}, \forall r \in (0,1). \quad (25)$$

On the other hand by Theorem 7 it holds

$$\lim_{r\to 1-0} F_r(\tau) = F^+(\tau), a.e. \tau \in \gamma,$$

and in result we have

$$\lim_{r\to 1-0} \|F_r(\tau)\|_X = \left\|F^+(\tau)\right\|_X, \; a.e.\ \tau \in \gamma.$$

Then applying Fatou's theorem we obtain

$$\left\|F^+(\cdot)\right\|_{L_{p-\varepsilon}(X)} \le \lim_{r\to 1-0} \|F_r(\cdot)\|_{L_{p-\varepsilon}(X)}, \forall\varepsilon \in (0, p-1).$$

From here follows

$$\left\|F^+\right\|_{L_{p)}(X)} \le \lim_{r\to 1-0} \|F_r\|_{L_{p)}(X)}.$$

Comparing with (25) we obtain the validity of (i).

Using (i), part (ii) is proved completely analogously to the classical case. For the completeness of presentation we give full proof. So, let $\{r_n\} \subset (0,1) : \; 0 < r_1 < r_2 < \ldots \to 1$, any sequence and $e \subset (0,1)$ any measurable set. Assume $E = (0,1) \setminus e$. We have

$$\left\|F_{r_n} - F^+\right\|_{L_{p)}(X)} \le \left\|\left(F_{r_n} - F^+\right)\chi_E\right\|_{L_{p)}(X)} + \|F_{r_n}\chi_e\|_{L_{p)}(X)} + \left\|F^+\chi_e\right\|_{L_{p)}(X)}. \tag{26}$$

As we know it holds

$$\lim_{n\to\infty} \|F_{r_n}(s)\|_X = \left\|F^+(s)\right\|_X, \;\; n \to \infty, \;\; \text{a.e. } s \in J.$$

Then we can apply the Egorov's theorem. Based on this theorem choose the set $E \subset (0,1)$ such that on E sequence $\{F_{r_n}(\cdot)\}$ uniformly converges to $F^+(\cdot)$. Consequently, it is evident that

$$\lim_{n\to\infty} \|F_{r_n}\chi_E\|_{L_{p)}(X)} = \left\|F^+\chi_E\right\|_{L_{p)}(X)}.$$

Then by part (i) we have

$$\lim_{n\to\infty} \|F_{r_n}\chi_e\|_{L_{p)}(X)} = \left\|F^+\chi_e\right\|_{L_{p)}(X)}.$$

From absolute continuity of the Lebesgue integral it follows that we can make the last two terms in (25) as small as desired.

Theorem is proved.

Consider the following Poisson –Bochner

$$(\mathscr{P}_r f)(t) = \frac{1}{2\pi}\int_{-\pi}^{\pi} P_r(t-s) f(s)\,ds,\ \ re^{it}\in\omega,$$

and Cauchy-Bochner

$$(\mathscr{K}_r f)(t) = \frac{1}{2\pi i}\int_{\gamma} \frac{f(\tau)}{\tau - re^{it}} d\tau,\ \ re^{it}\in\omega,$$

integrals. For Poisson-Bochner integrals we have proved the validity of the inequality (25):

$$\|\mathscr{P}_r f\|_{L_{p)}(X)} \le \|f\|_{L_{p)}(X)}, \quad \forall f\in L_{p)}(X), \quad \forall r\in[0,1).$$

Let $X\in UMD$ $\& f\in N_{p)}(X), \quad 1<p<+\infty$. Then by Theorem 7 we have

$$\lim_{r\to 1-0} (\mathscr{K}_r f)(t) = \left(R^+ f\right)\left(e^{it}\right),\ \text{ a.e. } t\in J,$$

where R^+ is a t-Riesz operator. Moreover, it is evident that for $\mathscr{K}_r f$ it is valid the following Poisson-Bochner formula

$$(\mathscr{K}_r f)(t) = \frac{1}{2\pi}\int_{-\pi}^{\pi} P_r(t-s)\left(R^+ f\right)\left(e^{is}\right) ds,\ \ \forall re^{it}\in\omega.$$

From here we immediately obtain

$$\|\mathscr{K}_r f\|_{L_{p)}(X)} \le \left\|R^+ f\right\|_{L_{p)}(X)} \le$$

$$\le \left\|R^+\right\|_{\left[N_{p)}(X)\right]} \|f\|_{L_{p)}(X)}, \quad \forall f\in N_{p)}(X), \quad \forall r\in[0,1).$$

Paying attention to the Theorem 11 we obtain the validity of the following

Theorem 12. *Let $X\in UMD$. Then there exists a constant $c_p>0$, depending only on X and p, such that for $\forall f\in N_{p)}(X), \quad 1<p<+\infty$ it hold:*

(i) $\|T_r f\|_{L_{p)}(X)} \le C_p \|f\|_{L_{p)}(X)}, \quad \forall r\in[0,1)$,

(ii) $\lim_{r\to 1-0} \|T_r f - R^+ f\|_{L_{p)}(X)} = 0$;

where $T_r\in\{\mathscr{P}_r;\mathscr{K}_r\}$ and $R^+\in\left[N_{p)}(X)\right]$ is t-Riesz operator.

Note that regarding the Bochner space $L_p(X)$ the analogous of this theorem was obtain in [24]. The same result regarding a separable space case was obtained a little earlier in [25] (for similar results see also the works [26, 27]).

3.6. Equivalent definition of ${}_nH^{\pm}_{p)}(X)$

In this subsection we define the grand Hardy –Bochner classes ${}_nH^{\pm}_{p)}(X)$ according to the classical way and prove equivalence of this definition previous one. We will consider only the case ${}_0H^{+}_{p)}(X) = H^{+}_{p)}(X)$. So, accept

Definition 4. *Assume*

$$\tilde{H}^{+}_{p)}(X) = \left\{ f \in \mathscr{A}^{+}_{\omega} : \|f\|_{\tilde{H}^{+}_{p)}(X)} < +\infty \right\},$$

where

$$\|f\|_{\tilde{H}^{+}_{p)}(X)} = \sup_{0<r<1} \|f_r(\cdot)\|_{L_{p)}(X)}, \quad f_r(t) = f\left(re^{it}\right).$$

Let $X \in UMD$. It is evident that spaces $H^{+}_{p)}(X)$ and ${}_0N^{+}_{p)}(X)$ are isometrically isomorphic and Poisson-Bochner operator $(\mathscr{P}_r \cdot)(t)$ realizes corresponding isomorphism. Moreover, for $f \in N_{p)}(X)$ it holds

$$f \in_0 N^{+}_{p)}(X) \Leftrightarrow \hat{f}_n = \int_{-\pi}^{\pi} f(t)\, e^{int} dt = 0, \quad \forall n \in N. \tag{27}$$

From $f \in \tilde{H}^{+}_{p)}(X)$ follows that $\exists f^{+} \in L_{p)}(X)$. In fact, $f \in \tilde{H}^{+}_{p)}(X) \Rightarrow f \in \tilde{H}^{+}_{p-\varepsilon}(X)$, $\forall \varepsilon \in (0, p-1)$. Then by results regarding $\tilde{H}_q(X)$, $q > 1$, (see. e.g. works [23, 24]) spaces we have that $\exists f^{+}(\tau)$, a. e. $\tau \in \gamma$. The fact that $f^{+} \in L_{p)}(X)$ it directly follows from definition and Fatou's theorem. Then for $f(\cdot)$ we obtain the Poisson –Bochner integral representation

$$f\left(re^{it}\right) = \frac{1}{2\pi} \int_{-\pi}^{\pi} P_r(t-s)\, f^{+}\left(e^{is}\right) ds, \quad \forall re^{it} \in \omega.$$

From this formula we obtain

$$\lim_{r \to 1-0} \|f_r(\cdot)\|_{L_{p)}(X)} = \left\|f^{+}(\cdot)\right\|_{L_{p)}(X)}.$$

Consider the question when for $f^{+}(\cdot)$ the relation

$$\lim_{r \to 1-0} \left\|f_r(\cdot) - f^{+}(\cdot)\right\|_{L_{p)}(X)} = 0, \tag{28}$$

is true ?

It is evident that if relation (28) holds, then $f^{+} \in N_{p)}(X)$, since $f_r \in C^{\infty}(X)$, $\forall r \in (0,1)$. From here direct follows that $f \in H^{+}_{p}(X)$ and at the same time $f^{+} \in_0 N^{+}_{p)}(X)$.

Conversely, let $f^+ \in N^+_{p)}(X)$. From $f \in \tilde{H}_p(X)$ follows that $f \in H_{p-\varepsilon}(X)$, $\forall\varepsilon \in (0, p-1)$. Then for $f(\cdot)$ we have Poisson-Bochner representation $f(re^{it}) = (\mathscr{P}_r f^+)(t)$. From here as previous case we obtain that the relation (28) holds. Moreover from $f \in H_{p-\varepsilon}(X)$ follows that the relation (28) holds and in result we obtain $f^+ \in_0 N^+_{p)}(X)$, consequently, $f \in H^+_{p)}(X)$. So, correspondence $F \to F^+$ (F^+- nontangential values) denote by $\theta : \theta F = F^+$. Therefore, we have proved that

$$\tilde{H}^+_{p)}(X) = H^+_{p)}(X) \Leftrightarrow \theta\left(\tilde{H}^+_p(X)\right) \in N_{p)}(X),$$

i.e. it is valid the following

Theorem 13. *Let $X \in UMD$ and the Hardy- Bochner classes $H^+_{p)}(X)$ and $\tilde{H}^+_{p)}(X)$ are defined by Definitions 3 and 5. Then it holds*

$$\tilde{H}^+_{p)}(X) = H^+_{p)}(X) \Leftrightarrow \theta\left(\tilde{H}^+_p(X)\right) \subset N_{p)}(X).$$

It is evident that $\theta\left(H^+_{p)}(X)\right) =_0 N^+_{p)}(X)$. Naturally the following question arises. Is it true $\theta\left(H^+_p(X)\right) =_0 N^+_{p)}(X)$?

We will show that this question has negative answer. In fact, take any function $f \in L_{p)}(X) \backslash N_{p)}(X)$ (such function exists, because $L_{p)}(X)$ is nonseparable). Consider the following series (formal for now)

$$f(\tau) = \sum_{n \in Z} \hat{f}_n \tau^n, \quad \tau \in \gamma, \tag{29}$$

where

$$\hat{f}_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)\, e^{-int} dt, \quad n \in Z.$$

Since, $f \in L_{p-\varepsilon}(X)$, $\forall\varepsilon \in (0, p-1)$, then by results of the work [37], the series (29) converges in $L_{p-\varepsilon}(X)$ to $f(\cdot)$, for $\forall\varepsilon \in (0, p-1)$ and in result it is evident that this series converges to $f(\tau)$ a.e. $\tau \in \gamma$. But it is obvious that this series does not converge in $L_{p)}(X)$ (since $f \notin N_{p)}(X)$) . Again by results of work [37] ($t-$ Riesz Property) the series

$$f^+(\tau) = \sum_{n=0}^{\infty} \hat{f}_n \tau^n, \quad \tau \in \gamma, \tag{30}$$

also converges in $f \in L_{p-\varepsilon}(X)$, $\forall\varepsilon \in (0, p-1)$.

Let's prove that $\exists f \in L_{p)}(X)$ such that $f^+ \notin N_{p)}(X)$. Really, let it not be so, i.e. $f \in L_{p)}(X) \Rightarrow f^+ \in N_{p)}(X)$. Also denote

$$f^-(\tau) = \sum_{n=-\infty}^{-1} \hat{f}_n \tau^n, \quad \tau \in \gamma. \tag{31}$$

Then from here follows that the series (30) also converges in $L_{p)}(X)$ and $f^- \in N_{p)}(X)$. Because, if $\exists f_0 \in L_{p)}(X)$ such that the series (31) does not converge in $L_{p)}(X)$ (i.e. $f_0^- \notin N_{p)}(X)$), then we can consider the function $\tilde{f}_0(t) = f_0(-t)$. It is evident that $\tilde{f}_0 \in L_{p)}(X)$ (it follows direct from the definition of $L_{p)}(X)$) . The corresponding series for $\tilde{f}_0(\cdot)$ is

$$\tilde{f}_0\left(e^{it}\right) = \sum_{n \in Z} \hat{f}_{-n} e^{int},$$

and consequently

$$\tilde{f}_0^+(t) = \sum_{n=0}^{\infty} \hat{f}_{-n} e^{int} = \sum_{n=-\infty}^{\infty} \hat{f}_n e^{-int} = \hat{f}_0 + f^-\left(e^{it}\right).$$

From here follows that $\tilde{f}_0^+ \notin N_{p)}(X)$ and this contradicts our assumption. Consequently, if for $\forall f \in L_{p)}(X) \Rightarrow f^+ \in N_{p)}(X)$, then $f^- \in N_{p)}(X)$. From here we direct obtain that $f = f^+ + f^- \in N_{p)}(X)$, i.e. $L_{p)}(X) = N_{p)}(X)$, and we get contradiction. In result, $\exists f_0 \in L_{p)}(X)$ such that $f_0{}^+ \notin N_{p)}(X)$, i.e. the corresponding series (29) does not converge in $L_{p)}(X)$. Consider the following Cauchy - Bochner type integral

$$F_0(z) = \frac{1}{2\pi i} \int_\gamma \frac{f_0(\tau)\, d\tau}{\tau - z}, \quad z \in \omega.$$

Since $f_0 \in L_{p-\varepsilon}(X)$, $\forall \varepsilon \in (0, p-1)$, then by results of the work [37] it holds $F_0^+(\tau) = f_0^+(\tau)$, a.e $\tau \in \gamma$ (where F_0^+ is non tangential values of F_0 on γ) . From here direct follows that $F_0^+ \notin H_{p)}^+(X)$ (since $F_0{}^+ \notin N_{p)}(X)$). Let's show that $F_0 \in \tilde{H}_{p)}^+(X)$.

In fact, it is evident that for $F(\cdot)$ it is valid the following $X-$ valued Sokhotski-Premelj formula

$$F_0^+(\tau) = \frac{1}{2} f_0(\tau) + (S f_0)(\tau), \text{ a.e. } \tau \in \gamma.$$

From here we immediately obtain that $F_0^+ \in L_{p)}(X) \Rightarrow f_0^+ \in L_{p)}(X)$. For $F_0(\cdot)$ it is valid the following Poisson - Bochner representation

$$F_0\left(re^{it}\right) = \frac{1}{2\pi}\int_{-\pi}^{\pi} P_r(t-s)\, f_0^+(s)\, ds, \qquad \forall re^{-it} \in \omega.$$

From here direct follows $F_0 \in \tilde{H}_{p)}^+(X)$. In result we have proved the validity of the following

Theorem 14. *Let $X \in UMD$ and the Hardy -Bochner classes $H_{p)}^+(X)$ and $\tilde{H}_{p)}^+(X)$ are defined by Definitions 3 and 5. The following strongly embedding is true $H_{p)}^+(X) \subset \tilde{H}_{p)}^+(X)$.*

3.7. Application to basicity

In this subsection we will apply the obtained results to the t-basicity of perturbed system of exponents. Let's accept the following

Definition 5. *Let $T^{\pm}(t) \in [X]$ a.e. $t \in J$, be some operators and $\{e_n^{\pm}\} \subset N_{p)}(J)$, $1 < p < +\infty$, be some system. We will say that the system*

$$\left\{T^+(\cdot)\, e_n^+(\cdot)\, ; T^-(\cdot)\, e_n^-(\cdot)\right\}_{n\in N},$$

forms a $t-$ basis for $N_{p)}(X)$ if for $\forall f \in N_{p)}(X)$ there exist unique sequences $\{f_n^{\pm}\}_{n\in N} \subset X$ such that

$$f(\cdot) = T^+(\cdot)\sum_{n=1}^{\infty} f_n^+ e_n^+(\cdot) + T^-(\cdot)\sum_{n=1}^{\infty} f_n^- e_n^-(\cdot),$$

in $N_{p)}(X)$.

Let $X \in UMD$. Then by results of subsection 3.1 the space $N_{p)}(X), 1 < p < +\infty$, has the following direct sum

$$N_{p)}(X) = N_{p)}^+(X) \dot{+} N_{p)}^-(X),$$

where $N_{p)}^{\pm}(X) = R^{\pm}\left(N_{p)}(X)\right)$; $R^{\pm} - t$-Riesz projectors. Denote

$$\theta_{p)}(X) = \left\|R^+\right\|_{[N_{p)}(X)]} + \left\|R^-\right\|_{[N_{p)}(X)]}. \tag{32}$$

Consider the following operator

$$T(t) = T^+(t)\, R^+ + T^-(t)\, R^-,$$

i.e. for $\forall f \in N_{p)}(X)$:

$$T(t) f(t) = T^{+}(t) f_{+}(t) + T^{-}(t) f_{-}(t), \quad t \in J,$$

where $f_{\pm}(t) = R^{\pm} f(t)$. Let $I \in [X]$ be an identity operator in X. Define $\Delta T^{\pm}(t) = I(t) - T^{\pm}(t)$, i.e. $\Delta T^{\pm}(t) f(t) = f(t) - T^{\pm}(t) f(t), \ \forall t \in J \& \forall f \in N_{p)}(X)$, where $I(t) \equiv I, \ \forall t \in J$. Denote by δ_T the following quantity

$$\delta_T = \max \left\{ \sup vrai \left\| \Delta T^{+}(t) \right\|_X ; \sup vrai \left\| \Delta T^{-}(t) \right\|_X \right\}. \tag{33}$$

Assume

$$\Delta T(t) = I(t) - T(t), \quad t \in J.$$

Note that $I(t), \ t \in J$, is the identity operator in $N_{p)}(X)$. Then it is evident that $I(t) = R^{+} + R^{-}, \quad t \in J$. Using this relation we have

$$\|\Delta T f\|_{L_{p)}(X)} = \left\| \Delta T^{+} R^{+} f + \Delta T^{-} R^{-} f \right\|_{L_{p)}(X)} \leq$$

$$\leq \left\| \Delta T^{+} R^{+} f \right\|_{L_{p)}(X)} + \left\| \Delta T^{-} R^{-} f \right\|_{L_{p)}(X)}.$$

Estimate

$$\|\Delta T^{\pm} R^{\pm} f\|_{L_{p)}(X)} = \sup_{0<\varepsilon<p-1} \left(\varepsilon \int_J \|(\Delta T^{\pm} R^{\pm} f)(t)\|_X^{p-\varepsilon} dt \right)^{\frac{1}{p-\varepsilon}} \leq$$

$$\leq \sup rai \|\Delta T^{\pm}(t)\|_X \|R^{\pm} f\|_{L_{p)}(X)}.$$

Consequently

$$\|\Delta T f\|_{L_{p)}(X)} \leq \delta_T \left(\left\| R^{+} f \right\|_{L_{p)}(X)} + \left\| R^{-} f \right\|_{L_{p)}(X)} \right) \leq \delta_T \theta_{p)}(X) \|f\|_{L_{p)}(X)},$$

$$\forall f \in N_{p)}(X) \Rightarrow \|\Delta T\|_{[N_{p)}(X)]} \leq \delta_T \theta_{p)}(X).$$

From here we obtain that if $\delta_T \theta_{p)}(X) < 1$, then the operator $T(t) = (I - \Delta T)(t)$ is invertible in $\left[N_{p)}(X)\right]$. In result we obtain that the equation

$$T^{+}(t) f_{+}(t) + T^{-}(t) f_{-}(t) = g(t), \quad t \in J,$$

for $\forall g \in N_{p)}(X)$ has a unique solution $(f_{+}; f_{-}) \in N_{p)}^{+}(X) \times N_{p)}^{-}(X)$.

Since the systems $\left\{e^{int}\right\}_{n \in Z_{+}}; \left\{e^{-int}\right\}_{n \in N}$ form a t-basis for $N_{p)}^{+}(X)$ and $N_{p)}^{-}(X)$, respectively, then from here we obtain that the system

$$\left\{ T^{+}(t) e^{int}; T^{-}(t) e^{-ikt} \right\}_{n \in Z_{+}; k \in N}, \tag{34}$$

forms t- basis for $N_{p)}(X)$. So, the following proposition is valid.

Proposition 7. *Let $X \in UMD\&T^{\pm}(t) \in [X]$, a.e $t \in J$, be some operators. If $\delta_T \theta_{p)}(X) < 1$, $1 < p < +\infty$, where the quantities $\theta_{p)}(X)$ and δ_T are defined by the expressions* (32) & *(*(33)*), then the perturbed exponential system* (34) *forms t- basis for $N_{p)}(X)$.*

Consider special case, when $T^{\pm}(t) = e^{\pm i\alpha(t)} I$, $t \in J$, where $\alpha \in L_{\infty}^{R}(J)$, be some function. Note that $\|R^{\pm}\|_{[N_{p)}(X)]} \geq 1$, and in result it is valid $\theta_{p)}(X) \leq \frac{1}{2}$. Paying attention to the relation

$$\left|1 - e^{\pm i\alpha(t)}\right| = 2\left|\sin\frac{\alpha(t)}{2}\right|,$$

we obtain

$$\delta_T = 2\left\|\sin\frac{\alpha(\cdot)}{2}\right\|_{L_{\infty}(J)}.$$

Consequently, if

$$||\alpha(\cdot)||_{L_{\infty}(J)} \leq 2\arcsin\frac{\theta_{p)}(X)}{2}, \tag{35}$$

then it is evident that it holds $\delta_T \theta_{p)}(X) < 1$. Thus, it is valid

Corollary 3. *Let $X \in UMD\&$ the function $\alpha \in L_{\infty}(J)$ satisfies the inequality* (35) *where the quantity $\theta_{p)}(X)$ is defined by* (32)*. Then the system*

$$\left\{e^{i(nt+\alpha(t) sign n)}\right\}_{n \in Z_+},$$

forms t-basis for $N_{p)}(X), 1 < p < +\infty$.

Let $\alpha(t) = \alpha t$, $t \in J$ where $\alpha \in R-$ some parameter. Then $||\alpha(\cdot)||_{L_{\infty}(J)} = \pi|\alpha|$ and according to the condition (35) we have also the following

Corollary 4. *Let $X \in UMD$ and it holds*

$$|\alpha| < \frac{2}{\pi}\arcsin\frac{\theta_{p)}(X)}{2},$$

where $\theta_{p)}(X)$ is defined by (32)*. Then the system*

$$\left\{e^{i(n+\alpha sign n)t}\right\}_{n \in Z_+},$$

forms t-basis for $N_{p)}(X), 1 < p < +\infty$.

Acknowledgements

This work was supported by the Azerbaijan Science Foundation. Grant No: AEF-MGC-2024-2(50)-16/02/1-M-02.

References

[1] P. Koosis, *Introduction to the theory of Hp spaces*. Mir, Moscow, 1984.

[2] K. Hoffman, *Banach Spaces of Analytic Functions*, Prentice-Hall, Englewood Cliffs, 1962.

[3] J. Garnett, *Bounded analytic functions*. Mir, Moscow, 1984.

[4] I.I. Danilyuk, *Nonregular boundary value problems on the plane*. Nauka, Moscow, 1975.

[5] G.M. Goluzin, *Geometric Theory of Functions of a Complex Variable*. Nauka, Moscow, 1966.

[6] B.T. Bilalov and Z.G. Guseynov, Basicity criterion for perturbed systems of exponents in Lebesgue spaces with variable summability. *Dokl. RAN*, v.436, No.5, 2011, pp.586-589.(*in Russian*)

[7] B.T. Bilalov and Z.G. Guseynov, Basicity of a system of exponents with a piece-wise linear phase in variable spaces. *Mediterr. J. Math.*, v.9, No.3, 2012, 487-498.

[8] B.T. Bilalov, A.A. Huseynli and S.R. El-Shabrawy, Basis Properties of Trigonometric Systems in Weighted Morrey Spaces. *Azerbaijan Journal of Mathematics*, v.9, No.2, 2019, pp. 183-209.

[9] B.T. Bilalov, Basis properties of some systems of exponentials, cosines, and sines. *Sibirsk. Mat. Zh.*, v.45, No.2, 2004, pp. 264-273; translation in *Siberian Math. J.*, v. 45, No.2, 2004, pp. 214-221.

[10] B.T. Bilalov, The basis property of some systems of exponentials of cosines and sines. *Differentsial'nye Uravneniya*, v.26, No.1, (1990, pp. 10-16; translation in *Differential Equations*, v.26, No.1, 1990, pp. 8-13.

[11] B.T. Bilalov, The Basis Property of a Perturbed System of Exponentials in Morrey-Type Spaces. *Siberian Mathematical Journal*, v.60, No.2, 2019, pp. 249-271.

[12] B.T. Bilalov, S.R. Sadigova and V.G. Alili, The Method of Boundary Value Problems in the Study of the Basis Properties of Perturbed System of Exponents in Banach Function Spaces. *Comput. Methods Funct. Theory*, v.24, 2024, 101–120.

[13] A.V. Bukhvalov, Hardy spaces of vector-valued functions. *Zap. Nauch. Sem. Leningrad Otdel. Mat. Inst. Akad. Nauk SSR*, v.65, 1976, pp. 5-16. (in Russian)

[14] O. Blasko, Hardy spaces of vector valued functions: Duality. *Trans. Of the Amer. Math. Soc.*, v.308, No. 2, 1988, pp. 495-507.

[15] J. Bourgain, *Vector valued singular integrals and H-BMO duality.* Probability theory and Harmonic analysis, Marcel Dekker, New York, 1986, pp. 1-19.

[16] E. Hille and R.S. Phillips, *Functional Analysis and Semi-Groups.* Amer. Math. Soc., 1996.

[17] A.S. Markus, *Introduction to the spectral theory of polynomial operator pencils.* American Mathematical Soc., 1988.

[18] L. Grafakos, *Modern Fourier Analysis*, Springer, 2009.

[19] N.P. Vekua, *Systems of singular integral equations.* Nauka, Moscow, 1970.

[20] N. Dunford and J.T. Schuartz, *Linear operators, General Theory.* Wiley-Interscience, 1988.

[21] T. Hytönen, J. Neerven, M. Veraar and L. Weis, *Analysis in Banach Spaces*, V.I, Springer, 2016.

[22] Bela Sz.-Nagy and F. Ciprian, *Harmonic Analysis of Operators on Hilbert Space.* North-Holland, Amsterdam and London, 1970

[23] A.V. Bukhvalov and A.A. Danilevich, Boundary properties of analytic and harmonic functions with values in a Banach space, *Math. Notes Acad. Sci. SSR*, v.31, 1982, pp. 104-110.

[24] A.A. Danilevich, Some boundary properties of abstract analytic functions, and their applications. *Mat. Sb. (N.S.)*, v.100(142), No.4(8), 1976, pp. 507-533; *Math. USSR-Sb.*, v.29, No.4, 1976, pp. 453-474.

[25] R. Ryan, Boundary values of analytic vector-valued functions. *Proc. Koninkl. Nederl. Acad. Ser. A.*, v.65, No.5, 1962, pp. 558-572.

[26] C. Grossetête, Sur certaines classes de fonctions harmoniques dans le disque a valeur dans un espace vectoriel topologique localement convexe. *C.R. Acad. Sci. Paris*, v.273, No.22, 1971, pp. 1048-1051.

[27] C. Grossetête, Classes de Hardy et de Nevanlinna pour les fonctions holomorphes à valeurs vectorielles. *C. R. Acad. Sei. Paris*, v.274, 1972, pp. 251-253.

[28] S. Bochner and A.E. Taylor, Linear functionals on certain spaces of abstractly-valued Functions, *Annals of Math.*, second series, v.39, No.4, 1938, pp. 913-944.

[29] D.R. Adams, *Morrey spaces*, Switzherland, Springer, 2016.

[30] V. Kokilashvili, A. Meskhi, H. Rafeiro and S. Samko, *Integral Operators in Non-Standard Function Spaces*, Volume 1: Variable Exponent Lebesgue and Amalgam Spaces, Springer, 2016.

[31] V. Kokilashvili, A. Meskhi, H. Rafeiro and S. Samko, *Integral Operators in Non-Standard Function Spaces*, Volume 2: Variable Exponent Holder, Morrey–Campanato and Grand Spaces, Springer, 2016.

[32] D.V. Cruz-Uribe and A. Fiorenza, *Variable Lebesgue spaces*, Birkhauser, Springer, 2013.

[33] Y. Sawano, G. Di Fazio and D.I. Hakim, *Morrey Spaces: Introduction and Applications to Integral Operators and PDE's*, Volume I, 2016.

[34] Y. Sawano, G. Di Fazio, D.I. Hakim, *Morrey Spaces: Introduction and Applications to Integral Operators and PDE's*, Volume II, 2020.

[35] C. Bennet and R. Sharpley, *Interpolation of operators.* Academic Press, 1988.

[36] R.E. Castillo, H. Rafeiro, *An Introductory Course in Lebesgue Spaces.* Springer Cham, 2016.

[37] B.T. Bilalov and S.R. Sadigova, The concept of t-basis and vector-valued Hardy classes. Turkish J. of Math. (accepted).

Bilal T. Bilalov
Odlar Yurdu University, Baku, Azerbaijan
Institute of Mathematics and Mechanics, The Ministry of Science and Education, Baku, Azerbaijan
Yildiz Technical University, Istanbul, Türkiye
Azerbaijan University of Architecture and Construction, Baku, Azerbaijan E-mail:
b_bilalov@mail.ru

Sabina R. Sadigova
Institute of Mathematics and Mechanics, The Ministry of Science and Education, Baku, Azerbaijan
Odlar Yurdu University, Baku, Azerbaijan
E-mail: s_sadigova@mail.ru

Yonca Sezer
Yildiz Technical University, Istanbul, Türkiye
E-mail: ysezer@yildiz.edu.tr

Received 07 October 2024
Accepted: 11 March 2025